Proof Style

Suppose we try to prove if A then B
by contradiction. We suppose A \land \neg B

Do some work

\[ \neg B \land \neg B \]

Should have done direct.

Do some work

\[ \neg A \land \neg A \]

Should have done contrapositive

Principal of Mathematical Induction (PMI)

Suppose we have a statement for all \( n \in \mathbb{N} \). Then, the statement is true for all \( n \in \mathbb{N} \), provided:

1) It is true for \( n = 0 \).

2) The statement being true for arbitrary k implies it is true for \( k+1 \), for all \( k \in \mathbb{N} \).

Proof By Induction

Basis: Show the statement is true for \( n = 0 \).

Induction step:

Induction hypothesis: Assume the statement is true for \( n \leq k \).

Then show the statement is true for \( k + 1 \).

The statement is true for all \( n \in \mathbb{N} \).

Principal of Strong Mathematical Induction (PSMI)

Suppose we have a statement for all \( n \in \mathbb{N} \). Then, the statement is true for all \( n \in \mathbb{N} \), provided:

1) It is true for \( n = 0 \).

2) The statement being true for all \( 0 \leq n \leq k \) implies it is true for \( k+1 \), for all \( k \in \mathbb{N} \).

Proof By Strong Induction

Basis: Show the statement is true for \( n = 0 \).

Induction step:

Induction hypothesis: Assume the statement is true for all \( 0 \leq n \leq k \).

Then show the statement is true for \( k + 1 \).

The statement is true for all \( n \in \mathbb{N} \).

- The Principal of Strong Mathematical Induction says that we can use all
previous $0 \leq n \leq k$ to show the statement is true for $n = k+1$.

- Colloquially speaking, it simply tells us that we have more tools at our disposal than the regular PMI, meaning proof by strong induction is “easier”

What if our base case is not $n = 0$? Here is a modified version of the PMI, where our base case is some arbitrary $b$.

**Principal of Mathematical Induction (arbitrary base case)**

Suppose we have a statement for all $n \geq b$. Then, the statement is true for all $n \geq b$, provided:

1) It is true for $n = b$.

2) The statement being true for arbitrary $k$ implies it is true for $k+1$, for all $k \geq b$.

**Proof By Induction (arbitrary base case)**

**Basis:** Show the statement is true for $n = b$.

**Induction step:**

**Induction hypothesis:** Assume the statement is true for $n \leq k$.

Then show the statement is true for $k+1$.

The statement is true for all $n \geq b$. 
Theorem

If $f$ is an integrable function and $\int_0^1 f(x)\,dx \neq 0$, then there exists a point $x$ on the interval $[0,1]$ such that $f(x) \neq 0$. 

IF $A$ then $B$
Theorem

A non-empty set $S$ is countable if and only if there exists an injective function $f: S \rightarrow \mathbb{N}$.

A iff B

If A then B

If B then A
Theorem
\sqrt{2} \text{ is irrational.}

\[ B \]

\text{If } \quad \text{then } \quad B

\text{implicit assumption}
Proposition
For all \( n \in \mathbb{N} \),
\[
1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2}
\]

Try \( n = 1 \)
\[
1 = \frac{1(1+1)}{2}
\]

\( \checkmark \)

Try \( n = 2, n = 3, \ldots, n = 99 \)
Try \( n = 100 \)
\[
1 + 2 + \ldots + 99 + 100 = \frac{100(100+1)}{2}
\]

From \( n = 99 \)
\[
\frac{99(99+1)}{2}
\]

We must show
\[
\frac{99(99+1)}{2} + 100 = \frac{100(100+1)}{2}
\]

\[5050 = 5050 \checkmark\]

Can we do this for \( n = k + 1 \)?
\[
1 + 2 + \ldots + k + (k+1) = \frac{(k+1)(k+2)}{2}
\]

We already know
\[
1 + 2 + \ldots + k = \frac{k(k+1)}{2}
\]
\[
\frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}
\]

LHS = RHS
Proposition
For all $n \in \mathbb{N}$,
\[
\frac{(2)(6)(10)(14) \cdots (4n - 2)}{(2)(6)(10)(14) \cdots (4n - 2)} = \frac{(2n)!}{n!}
\]

Basis step:
Show the statement is true for $n = 1$
\[
2 = \frac{(2 - 1)!}{1!}
\]
\[
2 = 2 \quad \checkmark
\]

Induction step:
Hypothesis: The proposition is true for arbitrary $n = k$.
\[
2 \cdot 6 \cdot 10 \cdots (4k - 2) = \frac{(2k)!}{k!}
\]
Multiply both sides by $4(k+1) - 2$
\[
2 \cdot 6 \cdot 10 \cdots (4k - 2)(4(k+1) - 2) = \frac{(2k)!}{k!} \cdot (4(k+1) - 2)
\]
\[
= (2k)! \cdot \frac{(4(k+1) - 2)(k+1)}{k!} \cdot \frac{k!}{(k+1)}
\]
\[
= (2k)! \cdot \frac{(4(k+1) - 2)(k+1)}{k!}
\]
\[
= \frac{(2(k+1))!}{(k+1)!}
\]
Proposition
For all $n \in \mathbb{N}$,

$$1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2}$$

Suppose there is a counter-example.
We pick the smallest one $x$.
We know $x-1$ is not a counter-example.

We agreed to finish this example next time...