Second-Order RRs

General 2nd-order RR

\[ a_n = s_1 a_{n-1} + s_2 a_{n-2} \]

We guess

\[ a_n = r^n \]

Subbing...

\[ r^n = s_1 r^{n-1} + s_2 r^{n-2} \]

\[ r^n - s_1 r^{n-1} - s_2 r^{n-2} = 0 \quad (\text{characteristic equation}) \]

Case: Two distinct roots

\[ a_n = c_1 r_1^n + c_2 r_2^n \]

Case: Repeated root

\[ a_n = (c_1 + c_2 n) r^n \]

Relations

A set of ordered pairs.
A relation from A to B is a set of ordered pairs where
- the first elements come from A
- the second elements come from B

Inverse
If R is a relation from A to B, then \( R^{-1} \) is a relation from B to A where the ordered pairs are reversed.

\[ (R^{-1})^{-1} = R \]

Functions
A relation \( f \) where \( (a, b) \in f \) and \( (a, c) \in f \) then it must be \( b = c \).

Notation:

\[ (a, b) \in f \]
\[ a \in A \]
\[ b \in B \]
\[ f(a) = b \]

Injection (one-to-one)
For all \( x, y \in A \) if \( x \neq y \) then \( f(x) \neq f(y) \).

Surjection (onto)
For all \( b \in B \) there exists an \( a \in A \) such that \( f(a) = b \).

**Bijection**

A function that is both an injection and surjection.

**Properties of bijections:**
1. \( f^{-1} \) is a function.
2. \( f^{-1} \) is a bijection.

We require \( \text{dom}(f) = A \).

**Counting Problems**

Suppose I have a box of Timbits (which I may or may not share).

Suppose I have:
- A total of \( n \) Timbits
- \( k \) friends I want to share with
- An appetite for \( j \) Timbits

Suppose all Timbits are unique/distinct (i.e., different flavours)...

How many different orders can I eat all \( n \) Timbits in?
\[ n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1 = n! \]

How many different orders can I eat \( j \) Timbits in?
\[ n \times (n-1) \times (n-2) \times \ldots \times (n-j+1) = \frac{n!}{(n-j)!} \]

How many different ways can I put \( j \) Timbits into a smaller box (order does not matter)?
\[ \frac{n!}{j!(n-j)!} = \binom{n}{j} = \binom{n}{j} \]
\[ a_n = 1 \cdot a_{n-1} + 6 \cdot a_{n-2} \]

Solution form

\[ a_n = r^n \]

We get

\[ r^n = r^{n-1} + 6r^{n-2} \]

\[ r^2 - r - 6 = 0 \]

\[ (r - 3)(r + 2) = 0 \]

\[ r = 3, -2. \]

Our solution is

\[ a_n = c_13^n + c_2(-2)^n \]

We will plug in \( n=0, n=1 \)

\[ a_0 = c_1 + c_2 \]

\[ a_1 = 3c_1 - 2c_2 \]

Do some algebra

\[ c_1 = \frac{2a_0 + a_1}{5} \]

\[ c_2 = \frac{3a_0 - a_1}{5} \]
\[ a_n = 6 \cdot a_{n-1} - 9 \cdot a_{n-2} \]

We guess: \( a_n < r^n \)

Sub in...

\[ r^n = 6r^{n-1} - 9r^{n-2} \]

\[ r^2 - 6r + 9 = 0 \]

\[ (r - 3)^2 = 0 \]

\[ r = 3 \]

So our solution is:

\[ a_n = (c_1 + c_2 n)3^n \]

Find \( c_1, c_2 \):

Try \( n = 0, n = 1 \)

\[ a_0 = c_1 \]

\[ a_1 = (c_1 + c_2)3 \]

Do some algebra...

\[ c_1 = a_0 \quad c_2 = \frac{a_1}{3} - a_0 \]
Proof

Prove the closed form solution to the recurrence relation

\[ a_n = s \cdot a_{n-1} + t \]

is

\[ a_n = \left( a_0 + \frac{t}{s-1} \right) s^n + \frac{-t}{s-1} \]

Proof By Induction!

Basis step: \( n = 0 \)

\[ a_0 = \left( a_0 + \frac{t}{s-1} \right) s^0 + \frac{-t}{s-1} \]

Induction step:

Assume the statement is true for \( k \)

\[ a_k = \left( a_0 + \frac{t}{s-1} \right) s^k + \frac{-t}{s-1} \]

We know

\[ a_{k+1} = s a_k + t \]

\[ a_{k+1} = s \left( \left( a_0 + \frac{t}{s-1} \right) s^k + \frac{-t}{s-1} \right) + t \]

Relation Examples

Let \( A = \{0,1,2,3\} \) and \( B = \{5,6,7,8,9\} \).

\[ \{(0.5), (1.5), (2.9), (2.7), (3.6)\} \]

\[ S = \{(0.6), (0.7), (0.9), (3.5)\} \]

\[ \{ \} \]

\[ S^{-1} = \{ (6,0), (7,0), (9,0), (5,3) \} \]
Function Examples

Let $A = \{0, 1, 2, 3\}$ and $B = \{5, 6, 7, 8, 9\}$.

\[
\begin{align*}
\{(0,5), (1,5), (2,9), (2,7), (3,6)\} & \quad \times \\
\{(0,6), (0,7), (0,9), (3,5)\} & \quad \times \\
\{} & \quad \times \\
F = \{(0,6), (1,6), (2,7), (3,9)\} & \quad \checkmark
\end{align*}
\]

Matrix

\[
\begin{bmatrix}
5 & 6 & 7 & 8 & 9 \\
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

Graph

A_0 \rightarrow A_1 \rightarrow A_6 \rightarrow A_7 \rightarrow A_8 \rightarrow A_9
Injection Examples

Let $A = \{0, 1, 2, 3\}$ and $B = \{5, 6, 7, 8, 9\}$.

- $\{(0,5), (1,7), (2,9), (3,6)\}$ ✓
- $\{(0,6), (1,9), (2,8), (3,6)\}$ ✗
Surjection Examples

Let $A = \{0, 1, 2, 3\}$ and $B = \{5, 6, 7\}.$

- $\{(0,5), (1,7), (2,6), (3,6)\}$
- $\{(0,6), (1,6), (2,7), (3,7)\}$
Bijection Examples

Let $A = \{0, 1, 2, 3\}$ and $B = \{5, 6, 7, 8, 9\}$.

$\{\{(0, 5), (1, 7), (2, 9), (3, 6)\}\}$

Not inj
Not surj

Let $A = \{0, 1, 2, 3\}$ and $B = \{5, 6, 7\}$.

$\{\{(0, 5), (1, 7), (2, 6), (3, 6)\}\}$

Not inj
Not surj

Let $A = \{0, 1, 2, 3\}$ and $B = \{5, 6, 7, 8\}$.

$\{\{(0, 5), (1, 7), (2, 6), (3, 8)\}\}$

Inj
Surj