Ice Cream

Cookie Dough (CD)  CD or V
Strawberry (S)       Many or Few
Chocolate (C)       C or V
Vanilla (V)         More or Less
                  S or V
                  Few or Many

\[ P = (X, \leq) \]
\[ X = \{ \text{CD, S, C, V} \} \]
\[ x \preceq y \text{ if we like } y \text{ more than } x \]
\[ V \preceq CD \]
\[ V \preceq C \]
\[ S \preceq V \]

Suppose we surveyed on CD vs C. What are the possibilities we could get?

\[ C \preceq CD \]
\[ CD \preceq C \]

Linear Extension

Let \( P = (X, \leq) \) be a poset.
A linear extension of \( P \) is a poset \( L = (X, \leq') \)
such that
\[ \forall x, y \in X, \quad x \leq y \Rightarrow x \leq' y \]

Theorem
Let \( P = (X, \leq) \) be a finite poset.
P has a linear extension.

"Proof"
If \( P \) is a lattice we are done.
Otherwise, \( P \) has incomparable elements.
In particular, call them \( x, y \).
From \( P = (X, \leq) \) we construct a new poset
\( L = (X, \leq') \) in the following way:
\[ \forall s, t \in X \]
if \( s \leq t \) then add \( s \leq' t \) to \( L \)
if \( s \leq y \) and \( x \leq t \) add \( s \leq' t \) to \( L \)
I claim that \( L = (X, \leq') \) is a poset that
extends \( P = (X, \leq) \).
The first part guarantees \( \leq' \) extends \( \leq \)
The second part guarantees \( y \leq' x \) in \( L \)
(take \( s = y \), \( t = x \))
\( L = (X, \leq') \) has one fewer incomparable pairs than \( P \).
Repeat...
$L = (X, \leq)$ will be a linear extension.

Realizers

Proposition
Let $P = (X, \leq)$ be a finite poset. Consider $x \neq y \in X$.
If $x \leq y$ in all linear extensions, then $x \leq y$ in $P$.
If $x \leq y$ in one linear extension and $y \leq x$ in another
then $x$ and $y$ are incomparable in $P$.

"Proof"

Contrapositive
Case 1: $x$ and $y$ incomparable
Case 2: $y \leq x$ in $P$

Suppose $x$ and $y$ comparable $(x \leq y)$ in $P$.
$x \leq y$ in all linear extensions.

How to reconstruct poset from all linear extensions?
Consider all pairs $(x, y)$ of elements
If $x \leq y$ in all linear extensions
then $x \leq y$ in $P$.
Otherwise $x$ and $y$ are incomparable in $P$. 
Show all of the linear extensions of the poset \( P = (2^{\{0,1\}}, \subseteq) \).

\[ X = \{ \emptyset, \{0\}, \{1\}, \{0,1\} \} \]
Show all of the linear extensions of the following poset.