There were some technical difficulties at the beginning of today’s class, and I was not able to connect my tablet to the projector in the classroom. So I ended up writing on the chalkboard during class.

Instead of directly posting what I wrote in class, I am posting the notes that I made when planning today’s class.
Graphs

The word "graph" has several connotations in English.

But we consider a specific definition of graph:

A graph is a pair $G = (V, E)$ where $V$ is a non-empty finite set and $E$ is a set of two-element subsets of $V$.

Ex

$G_1 = (\{1, 2, 3, 4, 5\},$
\{1, 2, 3\}, \{2, 3\}, \{1, 3\}, \{2, 4\}, \{4, 5\})$
Elements of \( V \) are called vertices (write \( V(G) \)).
Elements of \( E \) are called edges (write \( E(G) \)).

We have a nice visual way of representing graphs. Represent vertices as circles. Represent edges as lines between circles.

Ex

![Diagram](image)

Unlike Hasse Diagrams, position has no meaning.

Ex

![Diagram](image)

These two representations are equivalent.

**Adjacency**

Let \( G = (V, E) \) be a graph and \( u, v \in V \). We call \( u, v \) adjacent if \( \{u, v\} \in E \) (or neighbours).

Notation: \( u \sim v \)

Ex

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>5</td>
</tr>
</tbody>
</table>

**Neighbourhood**

Let \( G = (V, E) \) be a graph and \( v \in V \). The neighbourhood of \( v \) is

\[ N(v) = \{ u \in V : u \sim v \} \]

The set of all vertices adjacent to \( v \).
Ex \( N(2) = \{1, 3, 4\} \)
\( N(5) = \{4\} \)

Degree
Let \( G = (V, E) \) be a graph and \( v \in V \). The degree of \( v \) (written \( \text{deg}(v) \) or \( \text{dl}(v) \)) is the number of incident edges on \( v \).

Or, we can write \( \text{deg}(v) = |N(v)| \)

Ex \( \text{deg}(2) = 3 \)
\( \text{deg}(5) = 1 \)

Theorem
Let \( G = (V, E) \) be a graph. The sum of degrees of vertices of \( G \) is twice the number of edges

\[
\sum_{v \in V} \text{deg}(v) = 2|E|
\]

"Proof"
Each edge is incident on two vertices.
Each edge gets counted twice when summing incident edges over all vertices.

Complete Graph
Let \( G = (V, E) \) be a graph. We call \( G \) complete if for all \( u, v \in V \) with \( u \neq v \), \( \{u, v\} \in E \).

Every vertex is adjacent to every other vertex.

Ex The example we've been looking at is not complete.

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
\end{array}
\]

but this is complete.
We can represent a graph in set notation.
We can represent a graph using "circle and line" notation...
We can also represent a graph in matrix notation.

**Adjacency Matrix**
Create an \( n \times n \) matrix where \( n = |V| \)
Let the \( i^{th} \) row/column represent the \( i^{th} \) vertex
If \( i \sim j \) are adjacent
    put 1 in the \( i, j \) element
If \( i \not\sim j \) are not adjacent
    put 0 in the \( i, j \) element

**Example**

\[
\begin{pmatrix}
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

Adjacency matrices are always symmetric.

Recall, \( V \sim U \Rightarrow U \sim V \)
The main diagonal is always 0s.
(at least, with our current definition of graph)

**Subgraph**
Suppose \( G, H \) are graphs. \( G \) is a subgraph of \( H \)
if \( V(G) \subseteq V(H) \) and \( E(G) \subseteq E(H) \).

**Example**

\[
H = (\{1, 2, 3, 4, 5, 6\}, \\
\{(1, 2, 3), \{2, 3\}, \{2, 4, 3\}, \{2, 5, 3\}, \{3, 4, 3\}, \{3, 6\}, \\
\{4, 5\}, \{5, 6\}\})
\]
$G = (\{3, 4, 5, 6\}, \{\{3, 4\}, \{3, 6\}, \{4, 5\}, \{5, 6\}\})$

$G \cong H$