"Mental Arithmetic"

Me: Let’s warm up our brains. Somebody start the calculator app on their phone. Three people, please give me digits to make a number.

Class members: “4”, “7”, “3” (these were not the actual digits given ... I have forgotten them)

Me: Ok, now I’ll make a 6-digit number by repeating those, so we get 473473, Hmmmmm, that divides evenly by 11.

Calculator person: Correct!

Class: Wow, that’s amazing!

Me: Another one?

Another class member: 863 (or something like that)

Me: Ok, 863863 divides by ... 7

Calculator person: Yes it does.

Class: This is incredible!

Another class member: 587587 (I’m sure this wasn’t the number named, but it might have been)

Me: Um, 75 in the middle, 8 there, ... I think ... no ... ok, that divides by 91.

Calculator person: Yes!

Class: We’ve never seen such skills!
Me: This time, make it four digits.

Another class member: 3298 (or something)

Me: Ok, 2 in the second place, then 9 and 3 with 8 between them ... starts with 3 ... that divides by 137

Calculator person: Right!

Me: It also divides by 73

Class: Stop, stop, you are scaring us with your unearthly powers.

Me: Thank you, thank you, thank you for that tremendous applause.

Well it went something like that in my imagination, anyway.

The secret of this stunt - which you can use to either impress or annoy your family and friends - is based on a property of integers discovered by the Reverend James Booth in 1854 (see this paper by Osler and Kennedy). Booth noticed that any integer of the form $abcabc$ is divisible by 7, and divisible by 11, and divisible by 13.

There are several ways of proving this, but the easiest comes from the Osler & Kennedy paper: any number of the form $abcabc$ is equal to $abc \times 1001$.

Example: $283 \times 1001 = 283 \times 1000 + 283 = 283000 + 283 = 283283$

How does this help us? Well it turns out that 1001 is a composite number.

In fact, $1001 = 7 \times 11 \times 13$

So for any number of the form $abcabc$, we can write

$$abcabc = abc \times 1001 = abc \times 7 \times 11 \times 13$$

and in this form it is clear that $abcabc$ is a multiple of 7, and of 11, and of 13. Furthermore,
by combining these we can also see that \( abcabc \) is a multiple of 77 (\( 7 \times 11 \)), 91 (\( 7 \times 13 \)) and 143 (\( 11 \times 13 \))

Now, how about the \( abcdabcd \) number? Using the same type of observation, we can see that \( abcdabcd = abcd \times 10001 \) ... and since 10001 = 73*137 (thank you, Osler & Kennedy, for doing this work for us), we know immediately that \( abcdabcd \) divides by both 73 and 137.

So in class, when I appeared to be engaged in heroic feats of mental arithmetic I was in fact doing nothing more difficult than remembering 11, 7, 91, 137 and 73. I didn't have to pay any attention to the actual numbers being picked by the class - which is my excuse for not recalling them precisely here. I would have said “That divides by 11” etc. no matter what the actual numbers were.

If you decide to perform this, please remember to include the "thinking out loud" pretense of actually trying to do some mental arithmetic - it adds a lot to the effectiveness. If you just toss out the divisors instantly, people will guess that there is a trick to it (which of course there is). If you want to impress them, make it look like you are doing something difficult.

We didn't discuss this in class, but there are some simple but clever embellishments to this basic method. For example, if the number \( abcabc \) ends in 0, 2, 4, 6, or 8 then it divides by 2 - which is pretty obvious - but it also divides by 14 (\( 7 \times 2 \)), 22 (\( 11 \times 2 \)), and 26 (\( 13 \times 2 \)), and you look like a genius when you announce it divides by 154, 182 and 286.

Similarly, if it ends in 0 or 5 then it divides by 5 - which won't amaze anyone - but 35, 55, 65, 385, 455 and 715 are not obvious factors unless you know Booth's result.

One more extension: It may be less well known now than it used to be, but hundreds of years ago when I was a student I learned that if the digits of an integer sum to a multiple of 3, then the integer itself is a multiple of 3. For example, I can tell you immediately that 912342873111 is a multiple of 3 because its digits sum to 9+1+2+3+4+2+8+7+3+1+1= 42, and 42 = 3*14

So given an integer of the form \( abcabc \), we can quickly (and mentally) add \( a+b+c \). If that is a multiple of 3 then \( abcabc \) will also be a multiple of 3, so we can add 21, 33, 39, 231, 273 and 429 to the list of divisors of \( abcabc \).

Example: Suppose the number given is 870870. We can immediately state "That number divides by 154" (because it ends in an even number). When the applause dies down, we can add "It also divides by 385" (because it ends in 0). When the audience recovers, we can deliver the coup de grace "And, unless I have made an error, it also divides by 429." (because 8+7+0 = 15 which is a multiple of 3).
With an act like this, you are ready for Penn & Teller "Fool Us".

**Exercise:** I just claimed that if the digits of an integer sum to a multiple of 3, then the integer itself is a multiple of 3. **Prove this.**

The point of all this? Besides being fun and entertaining, it reminds us that mathematics is all about patterns. The better we are about recognizing the patterns, and seeing the mathematical relationships between things, the better equipped we are to understand our world and its complexities.

We reviewed the principle of Proof by Contradiction. PBC always follows the same template:

Claim : some statement P is true

Proof: Assume that P is false

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--- apply logical deductive arguments

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until we reach a contradiction - which consists of two statements that cannot both be true, such as "x is even" and "x is odd"

Therefore P is true

The first thing we need to talk about is why this is a valid proof.

It is valid because our mathematical discussions are always set in a world that is free of logical inconsistencies. That is, no statement within our mathematics can be both true and false at the same time. (Whether or not this is true in the real world is another question.) When we make an assumption, we are limiting our discussion to a subset of the mathematical universe - what I like to call a "hypothetical world" within the mathematical universe.

Some of these hypothetical worlds (or, if you like, "constrained situations") can actually exist within our mathematical universe (for example, the world defined by "Assume x is an even integer" is a well-defined subset of the universe) and some cannot (for example, the world defined by "Assume x ≠ x" does not exist, since a fundamental property of all mathematics is that each thing is equal to itself). For other assumptions, the situation is less obvious. For example, the world defined by "Assume the square root of 2 is a rational number" seems like it could possibly exist.
The main requirement for existence of a hypothetical world in mathematics is logical consistency. Since we require that the mathematical universe is logically consistent, we cannot accept the existence of a logical inconsistency in any part of it. If a hypothetical world defined by an assumption contains a logical inconsistency (a contradiction) then that hypothetical world cannot exist ... and that means that the assumption that defined it must be a false statement.

So in PBC, when we assume that statement P is false and then derive a contradiction, we are showing that "P is false" must be a false statement ... and that means "P is true" must be a true statement.

So let's do an example:

Claim: the sum of two even integers is always even

PBC: The first thing we need to do is make sure we know exactly what is being claimed. Mathematical statements like this are often tersely expressed because mathematicians all know how to interpret them. This statement could be written more explicitly as "for all integers x and y, if x is even and y is even, then x+y must be even".

This is important to know, because some beginners will try to prove the statement this way:

2 and 6 are even integers, and 2+6 = 8, and 8 is always even, so the claim is true. They misinterpret the phrase “two even integers” to mean “you get to choose two even integers” when it actually means “every possible pair of even integers”.

It is misunderstandings of this kind that lead people to try to "prove by example".

So the claim is a claim about all possible even integers. How do we negate this claim (which is what we need to do to start our PBC)? Again, this is where beginners often get into trouble.

The negation of "the sum of two even integers is always even" is NOT "the sum of two even integers is always odd" nor is it "the sum of two odd integers is always even" ... or several other possible substitutions of "odd" for "even". Since the claim is a "universal" claim, its negation is an "existence" claim. More precisely, since our claim says something is true for all pairs of even integers, its negation is the claim that there exists at least one pair of even integers where the statement is false.
So our PBC starts like this: Assume there exist even integers $x$ and $y$ such that $x+y$ is odd.

And now we can start working towards a contradiction.

- $x$ even $\Rightarrow x = 2^*t$ for some integer $t$ (this is the definition of 'even')
- $y$ even $\Rightarrow y = 2^*s$ for some integer $s$
- $x+y$ odd $\Rightarrow x + y = 2^*r + 1$ for some integer $r$ (this is the definition of 'odd')

$\Rightarrow 2^*t + 2^*s = 2^*r + 1$
$\Rightarrow 2^*(t+s-r) = 1$
$\Rightarrow 1$ is even
but $1 = 2^*0 + 1$, so $1$ is odd
$\Rightarrow 1$ is even and $1$ is odd ...... contradiction

Having arrived at a contradiction, we look back for the most recent assumption (in more complex proofs we may have made other assumptions along the way) - our contradiction tells us that the most recent assumption must be false. In this proof the only assumption is the initial one, so we know that even numbers $x$ and $y$ that satisfy the assumption do not exist. Therefore the original claim is true for all even numbers.

It may be of historical interest to know that there is a branch of formal logic called intuitionism (originated by L.E.J. Brouwer in the early part of the 20th century) which does not accept the validity of PBC. Intuitionists dispute the step from "if $P$ is false, then there is a contradiction" to "therefore $P$ must be true". The overwhelming majority of mathematicians recognize PBC as one of the most important and useful tools in mathematics.

Here is the example we did in class:

Claim: Let $x$ and $y$ be positive integers. Then $x^2 - y^2 \neq 1$

Let's think about how to negate this. We can certainly follow exactly the same reasoning as in the first example: since the claim asserts something is true for all positive integers, the negation is there exist positive integers where it is not true. This is exactly correct, so our
assumption is: assume there exist positive integers \( x \) and \( y \) such that \( x^2 - y^2 = 1 \)

We can also think of the claim as an implication:

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x \text{ positive and } y \text{ positive } \Rightarrow x^2 - y^2 \neq 1
\]

and now I want to take a little side trip to talk about implications and negations. An implication \( A \Rightarrow B \) should be understood to mean “whenever \( A \) is true, \( B \) is also true” (It most definitely does not mean “\( A \) causes \( B \)” ... \( A \) and \( B \) might both be caused by some other event or fact.) An equivalent interpretation is “It is impossible to have a situation in which \( A \) is true and \( B \) is false” ... and this is easy to negate. The negation of that statement is “It is possible to have a situation in which \( A \) is true and \( B \) is false”

It’s really just a version of something we already know: the negation of a universal statement is an existential statement. The implication states that something is univerally true, so its negation states that there exists at least one situation where it is false.

Many people wonder why the negation of \( A \Rightarrow B \) isn’t \( A \Rightarrow \neg B \). The answer is that this (suggested) negation is too strong. \( A \Rightarrow B \) claims that there is always a particular relationship between \( A \) and \( B \), so its negation simply says there is at least one case where that relationship does not hold. The statement \( A \Rightarrow \neg B \) goes further and claims that the relationship never holds.

A simple example: “I wear my hat on every Monday”. (Here \( A \) is “It is Monday” and \( B \) is “I am wearing my hat”. As an implication, it would be “It is Monday \Rightarrow I am wearing my hat”) The negation of this is not “I never wear my hat on Monday”. The proper negation is “There is at least one instance of a Monday on which I did not wear my hat”.
Back to $x$ and $y$. Recall we are trying to prove that $x$ positive and $y$ positive $\Rightarrow x^2 - y^2 \neq 1$

PBC: Assume there exist positive integers $x$ and $y$ such that $x^2 - y^2 = 1$

(A note on phrasing here: I was taught to use “such that” … but others use “where”, “with” or “and”. The key thing is that we are asserting three things:

- $x$ is a positive integer
- $y$ is a positive integer
- $x^2 - y^2 = 1$

As long as this is clear, the exact wording is not crucial.)

We can factor the left side to get $(x - y)(x + y) = 1$

From this point we can go in several directions to reach a contradiction. I’ll follow the path I used in class because it illustrates an important point.

We can see that $x$ and $y$ cannot be equal, because if $x = y$ then $(x - y) = 0$, which makes $(x - y)(x + y) = 1$ impossible. (Note: this is actually a tiny little PBC hidden in the middle of our larger proof.)

So we have two cases: either $x > y$ or $x < y$. We have to show that both of these lead to contradictions.

Case 1: $x > y$

In this case $(x - y) \geq 1$ and $(x + y) > 1$ … but this means $(x - y)(x + y) > 1$ which contradicts $(x - y)(x + y) = 1$

Case 2: $x < y$

In this case $(x - y)$ is negative and $(x + y)$ is positive, so $(x - y)(x + y) < 0$, so $(x - y)(x + y) < 1$, which contradicts $(x - y)(x + y) = 1$
Both cases lead to contradictions so we conclude that our assumption must be false. There are no positive integers \(x\) and \(y\) such that \(x^2 - y^2 = 1\), so we conclude that for all positive integers \(x\) and \(y\), \(x^2 - y^2 \neq 1\).