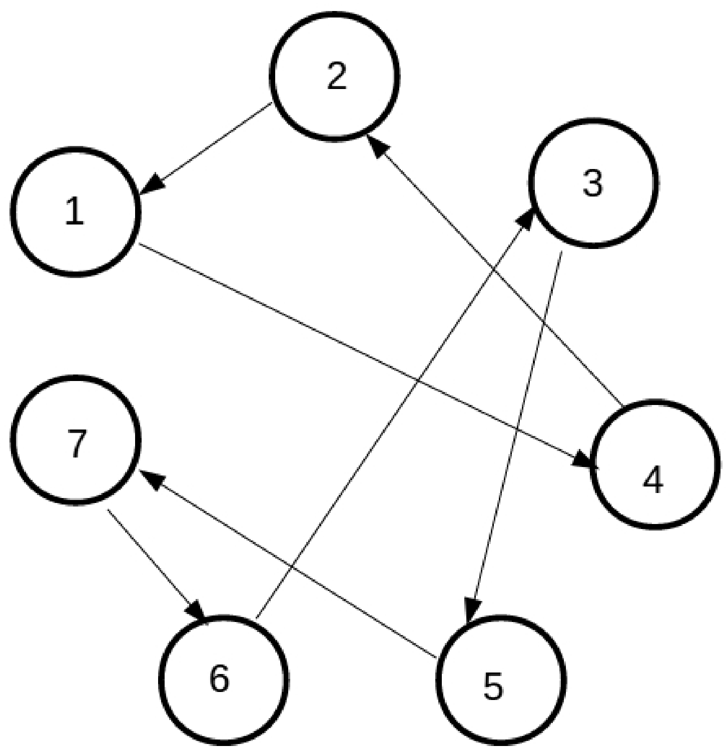
20200120

Picking up from the last day's notes … we are working with a permutation

$$\pi = (1,4,2)(3,5,7,6)$$

We can also represent this permutation by a diagram with each ordered pair represented by an arrow. The diagram for π looks like this:



So what can we do with a permutation expressed in cycle notation?

Computing the Inverse of a Permutation in Cycle Notation

Suppose we have a permutation π and we need to compute π^{-1}. We could do it from standard representation ...

For example, consider $\pi = \begin{bmatrix} 4 & 1 & 5 & 2 & 7 & 3 & 6 \end{bmatrix}$ To compute π^{-1}, we could see that "4" is in the first position, so the ordered pair (1,4) must be in π ... which means the ordered pair (4,1) must be in π^{-1}. Similarly, "2" is in the fourth position, so the ordered pair (4,2) is in π, so (2,4) must be in π^{-1}, and so on ... it's not that hard but it's a bit tedious.

But with the permutation expressed in cycle notation, computing π^{-1} is trivially easy: we just reverse each cycle. So the inverse of (1,4,2)(3,5,7,6) is simply (2,4,1)(6,7,5,3). You can check the details of this example to confirm that it works, but the logic of it is pretty straightforward: π contains the ordered pair (1,4) – this is encoded in the first cycle, so π^{-1} must contain the ordered pair (4,1) – and this is encoded in the reverse of the first cycle.

Note that **reversing** a cycle is very different from **rotating** a cycle: the cycles (a, b, c, d) and (c, d, a, b) represent exactly the same information, but (d, c, b, a) represents the inverse.

Since we can rotate the values within a cycle without changing the permutation, and we can list the cycles in any order, people sometimes ask if there is a "canonical" way to select the cycle representation of a permutation. To the best of my knowledge, there is no universally accepted canonical representation … but here is my personal preference: I rotate each cycle so that the lowest number in the cycle comes first, and I order the cycles so that the initial values in the cycles are in ascending order. So I would write the permutation $(7,4,9)(8,10)(6,1,5)(3,2)$ as $(1,5,6)(2,3)(4,9,7)(8,10)$

Composition of Two Permutations in Cycle Notation

Now suppose we have two permutations π and σ and we want to compute $\pi \circ \sigma$. (Remember, this means "the permutation that results when we apply σ, then apply π ") Once again cycle notation makes this very easy, and an example will show how this is done.

Let's use $\pi = \begin{bmatrix} 4 & 1 & 5 & 2 & 7 & 3 & 6 \end{bmatrix}$ and $\sigma = \begin{bmatrix} 4 & 7 & 6 & 1 & 5 & 3 & 2 \end{bmatrix}$.

In cycle notation, π = (1,4,2)(3,5,6,7) and σ = (1,4)(2,7)(3,6)(5) - you should check this.

(Why is 5 all by itself in σ? Because σ maps 5 to itself ... 5 forms a cycle of length 1.)

We can build the cycle notation for $\pi \circ \sigma$ as follows:

Start with 1. Apply σ to it, giving 4 (that is to say $\sigma(1) = 4$). Then apply π to that 4, giving 2 (that is, $\pi(4) = 2$). So in $\pi \circ \sigma$, we see that 1 $\rightarrow$ 2 (that is, $(\pi \circ \sigma)(1) = \pi(\sigma(1)) = \pi(4) = 2$).

So our first cycle in $\pi \circ \sigma$ starts (1, 2 ...)

Now let's see what $\pi \circ \sigma$ does to 2. σ takes 2 to 7, and π takes 7 to 6. So $(\pi \circ \sigma)(2) = 6$.

So our cycle in $\pi \circ \sigma$ now looks like (1, 2, 6 ...)

Now let's see what $\pi \circ \sigma$ does with 6. σ takes 6 to 3 and π takes 3 to 5, so $(\pi \circ \sigma)(6) = 5$.

The cycle in $\pi \circ \sigma$ we are building now looks like (1, 2, 6, 5 ...)

Following the same steps we see that $(\pi \circ \sigma)(5) = 7$. Then we discover that $(\pi \circ \sigma)(7) = 1$

So we have discovered that in $\pi \circ \sigma$, $1 \to 2 \to 6 \to 5 \to 7 \to 1$ is a cycle, and in cycle notation this is (1,2,6,5,7)

We still haven't dealt with 3 and 4. It doesn't matter which we start with, so let's start with 4. $\sigma(4) = 1$ and $\pi(1) = 4$, so $(\pi \circ \sigma)(4) = 4$. In cycle notation this is just (4)
You can verify for yourself that $(\pi \circ \sigma)(3) = 3$, giving (3) as the last cycle in $\pi \circ \sigma$

Putting all the cycles together, we get $\pi \circ \sigma$ = (1,2,6,5,7)(3)(4)

You should check this too!

Shorthand Notation for Cycles

When using cycle notation, sometimes we leave out the cycles of length 1. This can be ambiguous unless we know we are dealing with permutations from a particular S_n

In the previous example, if we specify that π and σ are in S_7 then we can write $\pi \circ \sigma = (1, 2, 6, 5, 7)$ and just leave out the (3) and (4). The reader will know they are cycles of length 1 because if they weren't we would have included them.

****** Bonus Material For Those Who Just Can't Say No to Permutations ******

This material will not be on any of our tests, but I hope some members of the class will find it interesting.

Consider these permutations of {1,2,3,4,5,6,7,8}:

$\pi = [2\ 3\ 1\ 5\ 6\ 4\ 8\ 7]$ and $\sigma = [4\ 7\ 6\ 8\ 3\ 5\ 2\ 1]$

When we put these into cycle notation, we get

$$\pi = (1,2,3)(4,5,6)(7,8) \text{ and } \sigma = (1,4,8)(2,7)(3,6,5)$$

They look very different ... and yet in a certain sense they are identical! Each has two cycles of length 3 and a single cycle of length 2. If you drew the diagrams for them as we saw at the beginning of this day's notes, then erased the labels, the two drawings would be interchangeable. Their essential structure is the same – we have just "re-labelled" the elements. Compare this to the permutation

$$\tau = [2\ 3\ 4\ 1\ 6\ 7\ 8\ 5] = (1,2,3,4)(5,6,7,8)$$

It's clear that τ has a fundamentally different structure from π and σ. This is a profoundly important concept that we will see again in other contexts. We say that \pi and \sigma are **isomorphic** – which means they have the same shape.

And what can we do with this? We can define an equivalence relation on permutations!

For permutations π and σ, both in S_n, we can say $\pi \sim \sigma$ iff π and σ have the same "cycle structure" . It should be clear that $\sim$ is reflexive, symmetric and transitive.

Now we can ask questions about this equivalence relation. For example, how many equivalence classes are there for S_n ? We can answer this by looking at the number of different cycle structures.

Let's consider permutations of [1 2 3]. The cycle structure of a permutation of this set must be either

(. . .) - a single cycle containing all three values, or
(.)(. .) - a cycle of length 1 and a cycle of length 2, or
(.)(.)(.) - three cycles of length 1 (there is only one permutation with this structure : ι)

and this means that in a very real sense, there are only 3 ways to permute a set with 3 elements.

Now consider permutations of {1, 2, 3, 4}. The possible cycle structures are

(. . . .)
(.)(. . .)
(.)(.)(. .)
(. .)(. .)
(.)(.)(.)(.)

so there are 5 ways to permute a set with 4 elements.

Unfortunately there is no simple formula to compute the number of equivalence classes of S_n for arbitrary values of n (there is a formula but it's **not** simple). It turns out that these are known as ***partition numbers***. Wikipedia has a good article about them :

https://en.wikipedia.org/wiki/Partition_(number_theory).

We can also use the cycle representation to prove this claim:

Let π be a permutation. Then there exists a positive integer k such that $\pi^k = \iota$

Proof: Suppose π contains a cycle c of length t. π^x where x is any positive multiple of t will restore the elements of c to their "natural" positions. Thus if π contains cycles with lengths in the set $L = \{l_1, l_2, \ldots l_m\}$ then $\pi^{lcm(l_1, l_2, \ldots l_m)} = \iota$

As an example, consider our old friend π = (1, 4, 2) (3, 5, 6, 7)

Consider what happens to 1 as we apply π repeatedly. We can see what happens very easily from the cycle notation

$\pi(1) = 4$
$\pi^2(1) = \pi \circ \pi(1) = \pi(\pi(1)) = \pi(4) = 2$
$\pi^3(1) = \pi \circ \pi^2(1) = \pi(\pi^2(1)) = \pi(2) = 1$
$\pi^4(1) = \pi \circ \pi^3(1) = \pi(\pi^3(i))) = \pi(1) = 4$
and we can see that the pattern repeats: for every k that is a multiple of 3, $\pi^k(1) = 1$
But the same is true for 4 and 2: for every k that is a multiple of 3, $\pi^k(4) = 4$ and $\pi^k(2) = 2$

But what about the values in the other cycle? We can see quickly that $\pi^4(3) = 3$, $\pi^4(5) = 5$, $\pi^4(6) = 6$ and $\pi^4(7) = 7$ … and the same will be true for any π^k where k is a multiple of 4

So if k happens to be a number that is a multiple of 3 *and* a multiple of 4, then π^k will map every value onto itself … in other words, $\pi^k = \iota$ for all common multiples of 3 and 4