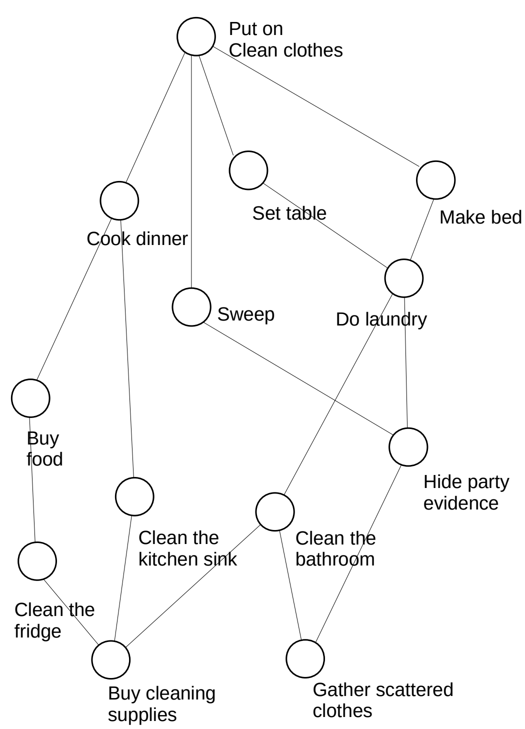
Orderings

We started our study of orderings by thinking about tasks involved in preparing for a visit from your mother (or someone else equally worthy of impressing). Here's a sample list of tasks:

Cook dinner
Put on clean clothes
Clean the kitchen sink
Clean the bathroom
Gather up scattered clothes
Make the bed
Sweep the floor
Do the laundry
Hide the evidence of wild partying
Buy cleaning supplies
Clean the fridge
Set the table
Buy food

Some of these tasks are unrelated, while others have a definite "this can't be done after that" relationship. We can arrange them in a diagram like this:



There are important things to understand about this diagram. First, the exact placement of the items is not critical – except that the lines between the items cannot be horizontal. A line which goes **upwards** from x to y indicates the relation "x cannot be done after y". We could indicate the same property by putting arrow-heads, but the "all lines go upwards, not downwards" convention works perfectly well.

If we think of these activities as a set, what we are doing is defining a relation on this set. A relation, of course, consists of a set of ordered pairs. This particular relation would contain pairs like ("Buy food", "Cook dinner") and ("Sweep","Put on clean clothes")

It is tempting to think that the relation we are defining is "must be done before" but we are actually going to stick with "cannot be done after" . Clearly it makes sense to say '"Buy cleaning supplies" cannot be done after "Clean the fridge" ' etc.

Note that there are many other related pairs that are implicit in this situation. For example, the "cannot be done after" relation must contain the pairs ("Buy cleaning supplies", "Cook dinner") and ("Gather scattered clothes", "Put on clean clothes") and many more.

Also, note that it is true to say "x cannot be done after x" for all x. So the pair ("Sweep","Sweep") must be in the relation … along with all other (x, x) pairs.

(Also note that as a curler, I have to attempt to work the phrase "Sweep, sweep" into as many contexts as I can.)

Finally, observe that if x and y are different activities in this set and the pair (x, y) is in the relation (ie. "x cannot be done after y") , then it is not possible to say "y cannot be done after x" … because that would make it impossible to do both x and y.

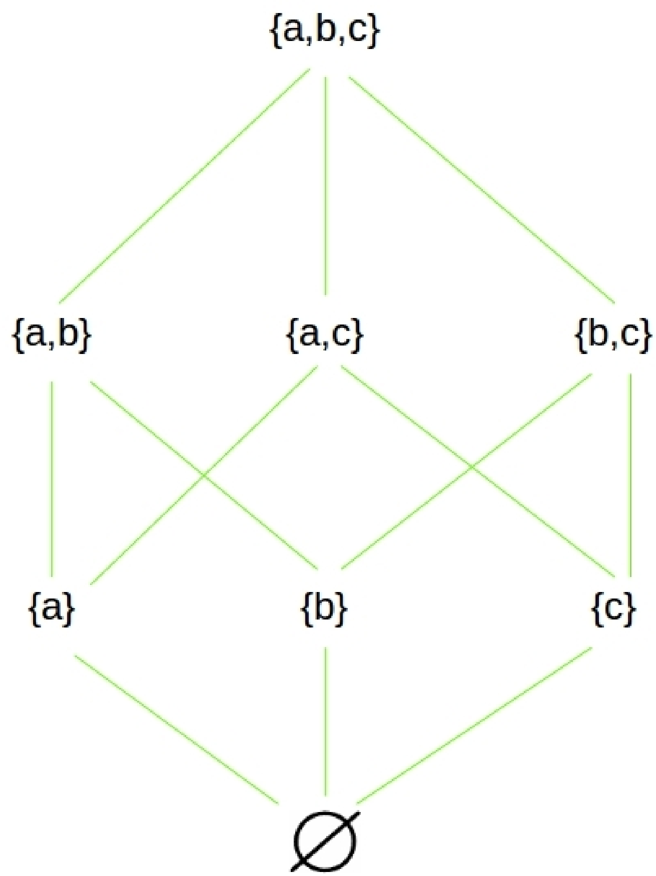
The last few paragraphs tell us that this relation is transitive, reflexive and anti-symmetric. We will see that this is not accidental.

Let's look at another example or two.

Let $X = \{a, b, c\}$ and let S = the set of all subsets of X

Ie $S = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

Consider the "is a subset of" relation on S. We can represent this relation by the same type of diagram as we used above:



Once again we need to be sure we understand exactly what is meant by this diagram. Each upward line connects a pair that satisfy the "is a subset of" relation. For example, $\{c\} \subseteq \{a, c\}$

Once again, there are lines missing from this diagram. For example $\{a\} \subseteq \{a, b, c\}$ but the diagram does not contain a line to indicate this … we will discuss why not after the next example.

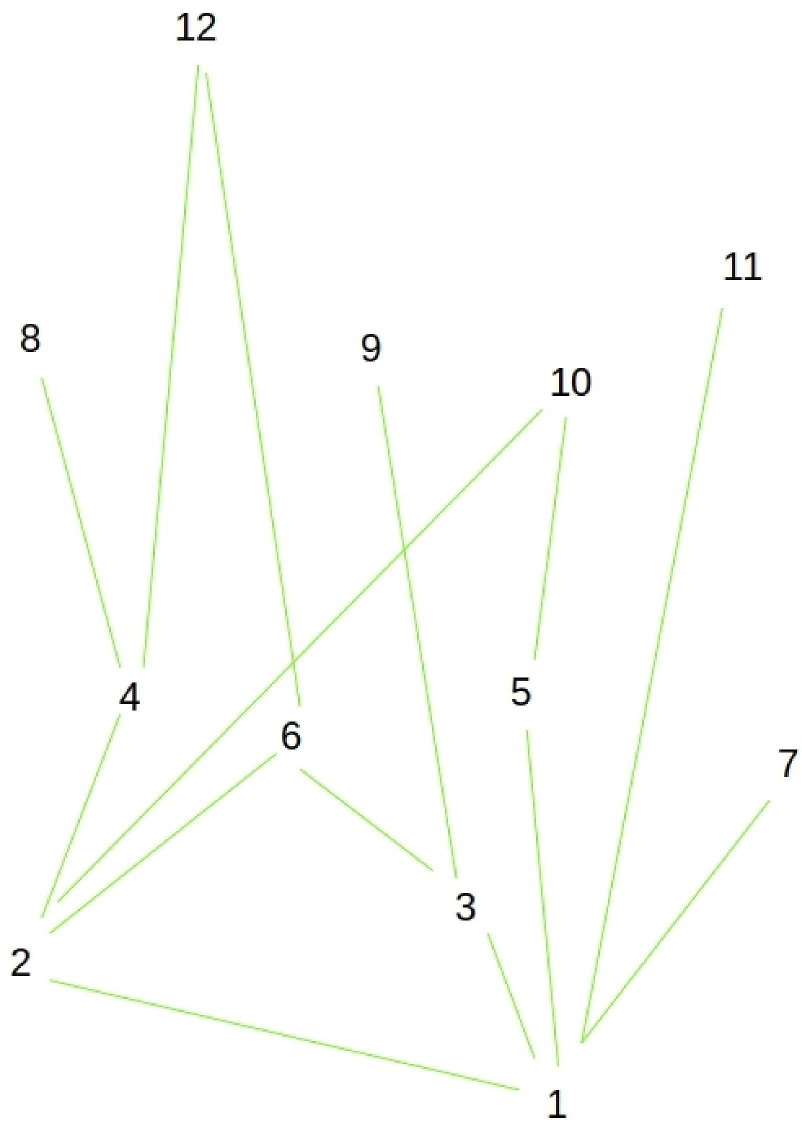
It is worth noting that the $\subseteq$ relation is reflexive, transitive and anti-symmetric … just like the "cannot be done after" relation.

A final (for now) example:

Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

and let R be the "divides" relation on S: $(x, y) \in R$ iff $x|y$

As with the other examples, we see that R is reflexive, transitive and anti-symmetric. We can represent this relation with a diagram like this:



As usual in these diagrams, a line from a up to b indicates that $(a, b) \in R$ … ie that $a|b$

Also, as in the earlier examples, there are some ordered pairs in the relation (such as $(2, 12)$) that are not represented by lines in the diagram.

Now let's look at the common properties of the three examples we have illustrated. Each of them consists of a set S and a relation R, where R has certain properties:

- **reflexive** – $(x, x) \in R \ \ \forall x \in S$
- **transitive** – if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$
- **anti-symmetric** – for any distinct values x and y, (x, y) and (y, x) cannot both be in R

We call such a relation a ***Partial Ordering***, or sometimes just a ***Partial Order***. We use the notation $P = (S, R)$ to mean that P is a partial ordering consisting of the set S and the relation R. We also refer to this as a ***partially ordered set***, sometimes reduced to ***poset.***

Why this name? Because the relation effectively puts the elements of S into some kind of order, but it doesn't necessarily provide ordering information about every pair of values in S.

For example, there is no ordering information about "Clean the fridge" and "Do the laundry". Neither one "cannot be done after" the other. Similarly $\{a\}$ is not a subset of $\{b, c\}$ and $\{b, c\}$ is not a subset of $\{a\}$, and neither $4|6$ nor $6|4$ is true.

Now, about those diagrams. They are called **Hasse Diagrams.** There's nothing really complicated or hard to understand about them – they are just a convenient way to represent posets. The goal when constructing a Hasse diagram is to provide complete information about the poset in the simplest manner, so we leave out any details that are either obvious or deducible. Remember that we know three things about the relation in a poset: it is reflexive, transitive and anti-symmetric. The diagrams are based on these principles:

(a) We will use the word "vertex" to generically denote the objects in the diagram. Each distinct object in S is represented by a vertex.

(b) Placement of vertices is irrelevant **except** that if $(x, y) \in R$ and $x \neq y$, the vertex for y **must be vertically higher** in the diagram than the vertex for x. This cannot lead to contradiction because we know R is anti-symmetric: if $x \neq y$, then we cannot have both (x, y) and (y, x) in R

(c) Lines (which we will often refer to as "edges") connect vertices that form ordered pairs in R. Because of Rule (b), there is no ambiguity about the meaning of the edges. If an edge joins x and y in the diagram then

- if y is higher in the diagram than x, then $(x, y) \in R$.
- if x is higher in the diagram than y, then $(y, x) \in R$.
- there is no third option: if x and y are related and $x \neq y$, they cannot be at the same vertical height in the diagram (because of Rule (b))

(d) We never include an edge that joins a vertex to itself, even though $(x, x) \in R \;\; \forall\, x \in S$. We know every vertex is related to itself because R is reflexive (R has to be reflexive because we are dealing with a poset) – so we don't need edges to remind us of this.

(e) We never include an edge whose existence can be deduced from other edges that we have included. This is where transitivity comes in. We know R is transitive, so if we know $(x, y) \in R$ and $(y, z) \in R$, we can deduce $(x, z) \in R$. We use the same idea in the diagram: if there is an edge from x to y and an edge from y to z, we can deduce the fact that $(x, z) \in R$ - we don't need an edge to tell us that.

Fortunately this all boils down to two very simple rules:

1. If $(x, y) \in R$ and $x \neq y$, then the vertex for y must be higher than the vertex for x

2. We put an edge between x and y if:
   $(x, y) \in R,\ \ x \neq y,$ and there is no z such that $(x, z) \in R$ and $(z, y) \in R$

So in the "divides" diagram, we don't put an edge from 2 to 12, even though $(2, 12) \in R$, because $(2, 4) \in R$ and $(4, 12) \in R$ (and similarly for many other potential edges).

In the "subset" diagram, we don't put an edge from $\varnothing$ to $\{a, b\}$ even though $\varnothing \subseteq \{a, b\}$, because $\varnothing \subseteq \{a\}$ and $\{a\} \subseteq \{a, b\}$ (and similarly for many other potential edges).

It should be clear that we can construct a Hasse diagram from every poset and that no matter how we arrange the vertices, the edges will be the same. Also, from any Hasse diagram we can reconstruct the corresponding poset.

Let's look at another example. This poset is defined using the ``refines'' relationship between partitions of a set:

Definition: a **partition** of a set is a collection of subsets such that each element of the set is in exactly one of the subsets. For example, if $S = \{a, b, c, d, e\}$ then $P = \{\{a, b\}\{c, e\}\{d\}\}$ is a partition of S ... one of many.

(Partitions are crucially important in software design – for example, we may have a collection of methods and objects that need to be grouped together into modules of a software system – each way of doing that is a different partition of the collection. Partitions are also essential in the field of data analytics. When faced with a huge amount of data, a common task is to subdivide the data into meaningful groups – sometimes called clusters – of similar items. Each such subdivision is a partition of the original data set.)

The number of partitions of a set with n elements grows very quickly as n gets large. In fact the question "How many different partitions are there of a set of size n?" is far beyond the scope of this course – but it is a fascinating topic and I encourage you to research it.

We are going to focus on a relation between partitions of a set. If P_1 and P_2 are partitions of the same set, then we say P_1 **refines** P_2 if each subset in P_1 is a subset of a subset in P_2

In notation, P_1 **refines** P_2 if $\forall X \in P_1 \quad \exists \quad Y \in P_2$ such that $X \subseteq Y$

If that sentence contained the word "subset" too many times, here is what it means: P_1 refines P_2 if each subset that belongs to P_2 can be formed by joining together one or more subsets of P_1. Equivalently, P_1 refines P_2 if the subsets that belong to P_1 can be formed by splitting apart some or all of the subsets that belong to P_2

For example, let $S = \{a, b, c, d, e, f\}$. Let $P_1 = \{\{a, c\}\{d, f\}\{b\}\{e\}\}$ and $P_2 = \{\{a, c\}\{b, d, e, f\}\}$ Here P_1 refines P_2

Another example: let S = {all species of animal life on Earth}. Let P_1 = { {all mammals} , {all non-mammals} } and P_2 = { {all mammals found only in Australia} , {all mammals that are not found only in Australia} , {all non-mammals} } . Here P_2 refines P_1

One more: let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
Let $P_1 = \{ \ \{1, 2, 3, 4, 5, 6, 7\}, \{8, 9, 10\} \ \}$ and
$P_2 = \{ \ \{2, 3, 5, 7\}, \{1, 4, 6\}, \{8\}.\{9\}, \{10\} \ \}$ Here P_2 refines P_1

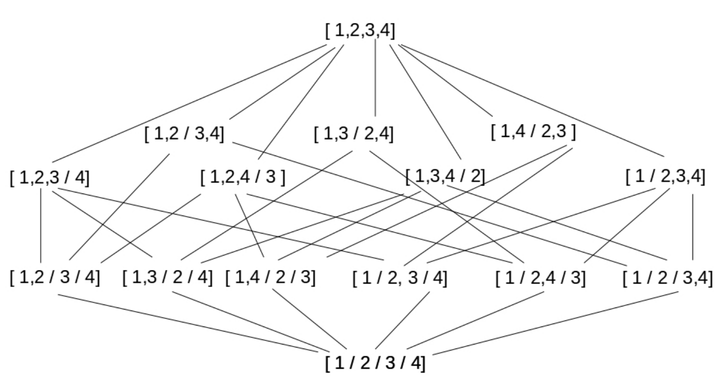
It should be clear that for any set S, "refines" is a relation defined on the set of all partitions of S. For any two partitions P_1 and P_2 we can determine whether P_1 refines P_2, or vice versa, or neither.

Note that since every set is a subset of itself, every partition is a refinement of itself.

We can use a simple notation to represent partitions – we just use / to separate the sets instead of all those { and } , and we put square brackets on the outside. Using this notation, the partitions in the first example above become P_1 = [a,c / d,f / b / e] and P_2 = [a,c / b,d,e,f]

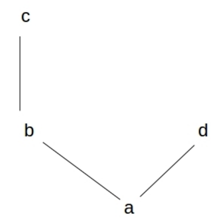
As an exercise, prove that P = (set of all partitions of a finite, non-empty set S,"refines") **is** a partial order.

Here is the Hasse Diagram for this poset when S = {1,2,3,4}.



This Hasse diagram is pretty dense with edges – you can see how difficult it would be to understand if we included all the implicit edges as well.

Now consider this Hasse diagram



It represents the partial order $\{(a,a),(b,b),(c,c),(d,d),(a,b),(a,d),(a,c),(b,c)\}$
We don't have to have a mathematical interpretation of the relation in order to work out the partial order associated with the Hasse diagram – it just has to be reflexive, transitive and anti-symmetric.