More Orderings

Linear Extensions

Given a finite poset $P = (X, \preceq)$, a natural question to ask is "can we find a linear order L that includes P?" It turns out that the answer is Yes, we always can. In fact we can go further.

(Why is this a natural question? Because one very common application of partial orderings is to represent the relationships between subtasks in a large project … either something like the Apartment Cleaning example, or something like a dependency diagram for a large software system. Frequently we need to process every item in the set, but we only have one processor (in the Apartment Cleaning example the processor was you, trying to get all the subtasks completed). We need to put the full set of items into a sequence so that the lone processor can work through them without ever trying to do a subtask before its predecessors have been completed.)

First, a definition: Let $P = (X, \preceq)$ be a poset. A **linear extension** of P is a linear order $L = (X, \preceq')$ where $\preceq \ \subseteq \ \preceq'$

The last three symbols on that line may look they can't possibly be meaningfully used in that way. But remember that a partial order is just a relation, and a relation is just a set of ordered pairs. So when we write $\preceq \ \subseteq \ \preceq'$, we mean " the set of ordered pairs that defines $\preceq$ " is a subset of " the set of ordered pairs that defines $\preceq'$ "

This definition of **linear extension** is our way of formalizing the concept of a linear order *including* a particular poset. If L is a linear extension of P, then L includes all the relationships (that is to say, the ordered pairs) in P, and also enough new relationships (ordered pairs) to eliminate all incomparabilities.

In class I planned to present an informal description of how we extend a partial ordering to a linear ordering by "squooshing" the chains together. It's easy to visualize: push the sides of the Hasse Diagram together until all the vertices are in a vertical line. Now put in the lines required to make a linear ordering of the vertices from bottom to top. Since squooshing is not a well-defined mathematical operation, I'll be a bit more rigourous in these notes – but the concept remains the same: if x and y are not in a chain together, we can squoosh them together with x above y **or** with y above x.

Theorem: Let P = (X, $\preceq$) be a finite partial order in which x and y are incomparable. Then there exists a linear extension L_1 of P such that in L_1, x < y. There also exists a linear extension L_2 of P such that in L_2, y < x

The proof of this simple theorem looks scary – there are tables and cases and strange symbols. But it's really just working through a bunch of very simple arguments.

Proof: Let P = (X, $\preceq$) be a finite partial order in which x and y are incomparable (that is, neither x $\preceq$ y nor y $\preceq$ x)

We define a new partial order $\preceq'$ on X as follows:

x $\preceq'$ y

if a $\preceq$ b then a $\preceq'$ b

if a $\preceq$ x and y $\preceq$ b then a $\preceq'$ b

Ok, $\preceq'$ certainly includes x $\preceq'$ y, and it includes all of $\preceq$ ... but is it a partial order?

Reflexive: $\preceq'$ is reflexive because $\preceq$ is reflexive

Anti-symmetric: Suppose $a \preceq' b$ and $b \preceq' a$. We need to show $a = b$

$a \preceq' b$ can be true in two ways: either $a \preceq b$ or $a \preceq x$ and $y \preceq b$

$b \preceq' a$ can be true in two ways: either $b \preceq a$ or $b \preceq x$ and $y \preceq a$

This gives us four cases to consider in order to show that $a = b$

	$b \preceq a$	$b \preceq x$ and $y \preceq a$
$a \preceq b$	Case 1	Case 2
$a \preceq x$ and $y \preceq b$	Case 3	Case 4

Case 1: $a \preceq b$ and $b \preceq a$: In this case $a = b$ because $\preceq$ is anti-symmetric

Case 2: $a \preceq b$ and $b \preceq x$ and $y \preceq a$: we can combine these three facts to get $y \preceq a \preceq b \preceq x$ from which we see $y \preceq x$ ... but this is impossible since x and y are incomparable. Thus this case cannot exist.

Case 3: $b \preceq a$ and $a \preceq x$ and $y \preceq b$: as for Case 2, this case cannot exist .

Case 4: $a \preceq x$ and $y \preceq b$ and $b \preceq x$ and $y \preceq a$: from $y \preceq a$ and $a \preceq x$ we get $y \preceq x$, which is impossible ... so this case cannot exist either.

Thus only Case 1 is possible, and in Case 1 we saw $a = b$

Therefore $a = b$, and therefore $\preceq'$ is anti-symmetric.

Proof that $\preceq'$ is transitive follows a similar pattern. (**Exercise: complete this part of the proof.)**

Thus $\preceq'$ is a partial order on X, and it contains ***fewer*** incomparable pairs than $\preceq$ does. If $\preceq'$ is a linear order (no incomparable pairs at all) then we have extended $\preceq$ to a linear order, as required. If $\preceq'$ is *not* a linear order, we can choose an incomparable pair w and z and extend $\preceq'$ to $\preceq''$ ... and if $\preceq''$ is not a linear order, we can extend it to $\preceq'''$, etc. Eventually we must eliminate all incomparable pairs, ending up with a linear order that contains the ordered pair (x,y) as stated in the theorem.

Now by switching x and y throughout all of the preceding argument, we will get a linear order that contains (y,x) as an ordered pair.

Thus the theorem is proved.

Dimension

As we now know, every poset that is not a linear order can be extended to at least two different linear extensions.

Observation: Let P = (S, $\preceq$) be a poset. Let x and y be distinct elements of S. If x $\prec$ y, then x precedes y in ***all*** linear extensions of P. Conversely, if x and y are incomparable elements of S, there exists at least one linear extension of P in which x $\prec$ y, and at least one linear extension of P in which y $\prec$ x. Our proof of the previous theorem justifies this observation.

This suggests an interesting question: if we are given the set of all linear extensions of a poset, can we reverse-engineer it and determine the original poset?

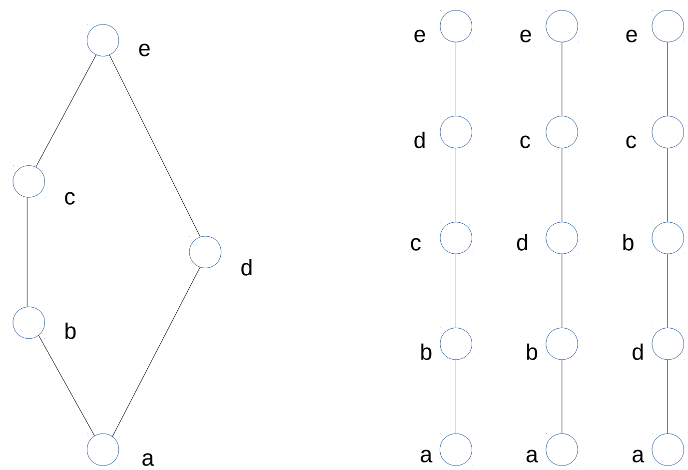
The answer is yes!

For each pair of distinct items x and y, we can check to see if x precedes y in **all** the linear extensions ... if so, we know x $\prec$ y. Similarly if y precedes x in **all** the linear extensions, we know y $\prec$ x. However, if there is at least one linear extension in which x precedes y, and at least one in which y precedes x, then x and y are incomparable. Thus we can determine exactly which pairs are related, and how they are related.

The next question is: do we always need the full set of linear extensions of a poset to reconstruct the poset?

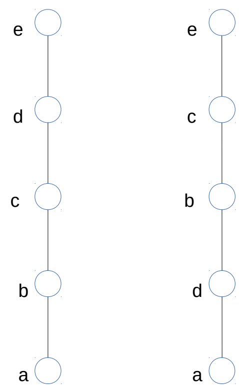
The answer (which surprised me the first time I learned it) is no!

Consider this example:



The poset on the left has exactly three linear extensions – they are shown on the right. But we can reconstruct the poset using just the first and third linear extensions!

Here they are:



To see that these give all the information needed, let's look at b and c. We see that b precedes c in both linear extensions, so we know b $\prec$ c. We can deduce the same thing for pairs such as d and e, a and b, etc.

But consider b and d. B precedes d in one of the linear extensions, and d precedes b in the other. Thus we know b and d are incomparable in P. Similarly we can discover that c and d are incomparable in P.

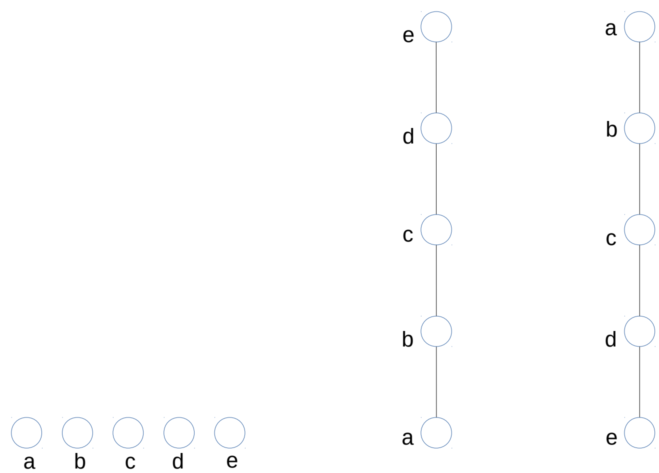
Definition: If P is a poset and R = {$L_1, L_2, \ldots, L_i$} is a set of linear extensions of P that allows us to completely reconstruct P, we call R a **realizer** for P.

Note that every poset has at least one realizer since the full set of linear extensions of any poset is a realizer for that poset.

Definition: We have seen that for a given poset, some realizers are smaller than others. Let P be a poset and let R be a realizer for P with the property that there is no smaller realizer for P. If the number of extensions in R is k, then we say that P has **dimension** k (which we sometimes write as **dim P = k**)

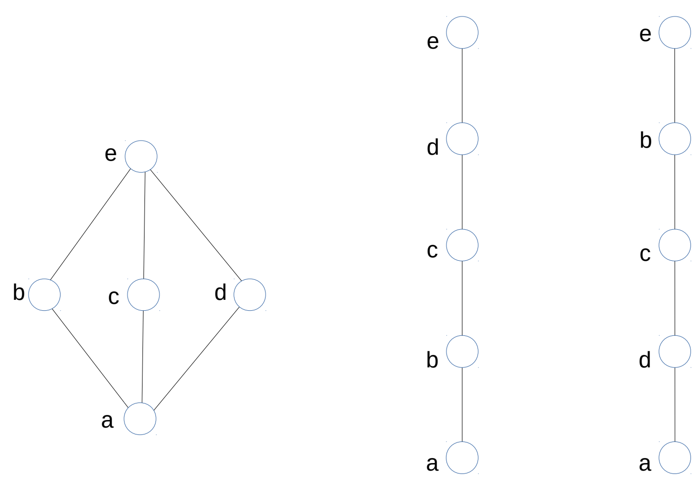
For the poset in the example above, we can see there is no realizer of size 1 (because we need at least two extensions to show that b and d are incomparable). Since we have found a realizer of size 2, we now know that the dimension of this poset is 2.

So, how high can the dimension of a poset be? Consider this example:



The poset is shown on the left ... none of the elements are related to each other.   Since *any* linear order of {a,b,c,d,e} is a valid linear extension of this poset, we see that there are 5! = 120 linear extensions of this poset. And yet the two extensions on the right in the diagram are a realizer for the poset! (Make sure you see why). So once again, this poset has dimension 2.

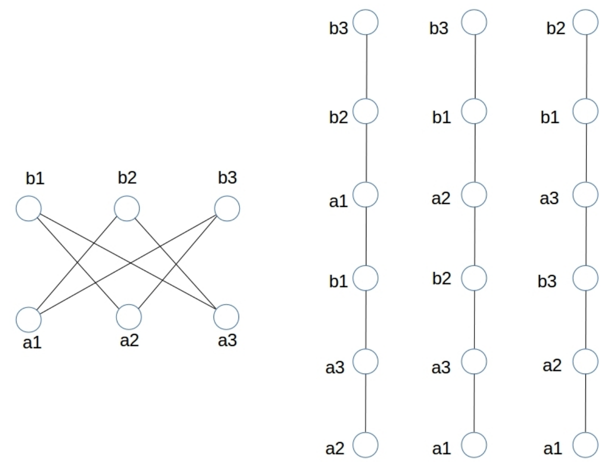
Another example:



Again we see that the two linear extensions on the right form a realizer for the poset, (and again we see that the dimension of the poset must be > 1 since there are incomparable elements) so we see this poset also has dimension 2.

Perhaps you are forming an hypothesis that all posets have dimension = 1 (which is the case if the poset is a linear order) or dimension = 2 ... which is the case for all the other posets we have looked at.

Sadly the situation is more complicated than that. Consider *this* example:



I claim that the poset shown here has dimension at least 3. We need at least one linear extension that has b1 preceding a1 ... but in this linear extension we **must** have a2 preceding b1 which precedes a1 which precedes b2 … so a2 precedes b2. Similarly, this linear extension must have a3 preceding b3. Therefore we need another linear extension with b2 preceding a2 (so that we can tell they are incomparable in the partial ordering) ... and in this linear extension as in the first one, a3 must precede b3. Therefore the realizer must contain a third linear extension with b3 preceding a3 (since these elements are incomparable, a realizer must contain at least one linear extension with a3 below b3, and at least one linear extension with b3 below a3).

Three such linear extensions are shown in the diagram, and we can see that these form a realizer for this poset. Therefore this poset has dimension 3.

We can generalize this example to have elements a_1 to a_n and b_1 to b_n - these posets are called the **standard examples** - the one shown above is called S_3 because it has 3 elements in each of its two subsets.

It turns out that the standard example of size n (ie. S_n) has dimension n.

Thus there is no limit to how large the dimension of a poset can be.

Let's consider one more question – in some ways, the most interesting of all: Given a poset P, can we determine its dimension?

If P is a linear order, dim P = 1

We can determine if dim P = 2 using an algorithm that is (unfortunately) outside the scope of this course. This algorithm runs fairly quickly. You can learn more about this here: https://onlinelibrary.wiley.com/doi/abs/10.1002/net.3230020103 but be prepared to learn a lot of terminology!

But beyond that ... nobody knows if there exists any efficient way to answer the question "Does dim P = k?" where k is any integer ≥ 3. In fact, this has been identified as one of the most difficult questions to answer in all of computational mathematics.

This is a tiny foretaste of a topic that will be explored in future courses. The world of problems that we can solve using computers seems to split into two classes: problems that we can solve quickly, and problems that are so difficult that we don't believe we will **ever** be able to solve them quickly. And sometimes the difference between the two classes is as simple as changing a 2 to a 3. People have spent decades trying to figure out why.