Subgraphs

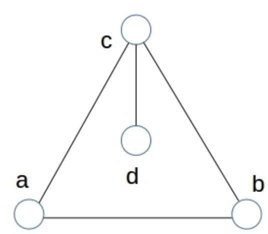
As you might guess, a subgraph is part of a graph (the relationship between graphs and their subgraphs is almost identical to the relationship between sets and their subsets).

Formally, let G = (V,E) and H = (X,F) be graphs. If $X \subseteq V$ and $F \subseteq E$ then H is a **subgraph** of G

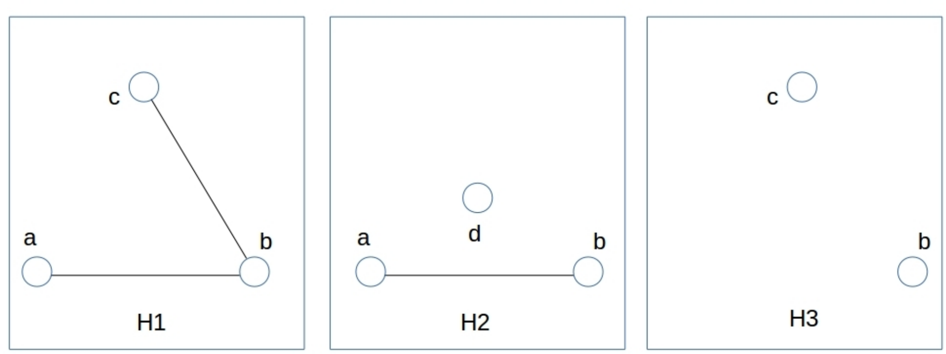
In plain words, this means H is a subgraph of G if every vertex and edge of H is also in G.

A subgraph H of G can contain some, all, or none of the edges of G that join the vertices in H.

For example, consider this graph G. Each of the graphs H1, H2 and H3 is a subgraph of G – and there are many many more subgraphs not shown here. As a practice problem, compute the total number of subgraphs of this graph.



G



Note that a subgraph is not allowed to "invent" edges that were not in G, so for the graph shown above, no subgraph can contain an edge joining ***a*** and ***d*** or an edge joining ***b*** and ***d***

There is something fundamentally different about the subgraph H2 in the diagram, that distinguishes it from H1 and H3. In both H1 and H3, potential edges that could have been in the subgraph have not been included. For example, the edge {a,c} is in G and both these vertices are in H1, so this edge *could have been* included when H1 was defined. Similarly, given that H3 contains the vertices b and c, the edge {b,c} *could have been* included when H3 was defined.

But the edge set of H2 contains *every allowable edge* between the vertices a, b and d (remember we are only allowed to use edges that are in G).

A subgraph that contains every allowable edge between its vertices is called an **induced subgraph.** An induced subgraph is completely determined by its vertex set – once we have decided which vertices are in the subgraph H, the edges of H are exactly the edges of G that join the chosen vertices.

Formally, if H = (Z,F) is a subgraph of G = (V,E) and $F = (Z \times Z) \cap E$, then H is an induced subgraph of G.

Question: If G is a graph with n vertices, how many induced subgraphs does G have?

One more definition regarding subgraphs: Let G be a graph and let H be a subgraph of G. If the vertex set of H is exactly the vertex set of G, then H is called a **spanning subgraph** of G.

Spanning subgraphs are particularly important when we are using graphs to represent communication or transportation networks. If each vertex represents a city on a map and the edges represent highways, we often need to explore subgraphs that include all the cities but possibly leave out some of the highways ... these are spanning subgraphs!

Question: If G is a graph with n vertices, how many induced spanning subgraphs does G have?

Note that

- every graph is a subgraph of itself (so the "subgraph of" relation is reflexive),
- if H is a subgraph of G, and G is a subgraph of F, then H is a subgraph of F (so the "subgraph of" relation is transitive)
- if H is a subgraph of G and G is a subgraph of H, then G and H are the same graph (so the "subgraph of" relation is anti-symmetric)

So the "subgraph of" relation is a **partial order**!