SOLUTIONS

Student Number (Required) ______________________

Name (Optional) ________________________________

This is a closed book test. You may not refer to any resources.

This is a 50 minute test.

Please write your answers in ink. Pencil answers will be marked, but will not be reconsidered after the test papers have been returned.

The test will be marked out of 50.

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General marking philosophy: a student who gives enough of an answer to show they understood what they were supposed to do, even if they couldn’t do it (or made lots of errors while doing it) should get at least 50% on that question.

Full marks should be given if a solution is sound and not missing anything important.

Feel free to give marks like 9.5/10 to a solution that is correct but contains a minor error.

A student should only get 0/10 on a question if they made no attempt to answer it at all.
Question 1 (10 marks)

Let $a$ and $b$ be integers, such that $a$ is even, and $a+b$ is odd.

Using Proof by Contradiction, prove that $b$ is odd.

(Hint: any even integer $x$ can be written as $2^k$ for some integer $k$)

Solution:

Since $a$ is even we know $a = 2^x$ for some integer $x$

Suppose $b$ is even.

Then $b = 2^y$ for some integer $y$

=>$a+b = 2^x + 2^y$

=$2^{x+y}$

=>$a+b$ is even

CONTRADICTION to premise that $a+b$ is odd

Therefore our supposition that $b$ is even is false. Therefore $b$ is odd.

Marking Question 1:

There are many other valid sequences of steps. The important thing is to start by assuming the opposite of what you want to prove, then get to a contradiction. If the student shows that they understand that, they should get at least 6/10
Question 2 (10 marks)

Prove that for all $k \geq 1$, the sum of the first $k$ even integers is $k^2 + k$

Use proof by induction.

Example: consider the sum of the first 3 even positive integers.

\[ 2 + 4 + 6 = 12, \text{ and } 3^2 + 3 = 12 \]

Here is a useful way of writing the general equation you need to prove:

\[ \sum_{i=1}^{k} (2i) = k^2 + k \]

Solution:

**Base case:** $n = 1$. The sum of the first 1 even integer is 2, and $1^2 + 1 = 1+1 = 2$

**Inductive step:**
Assume that for some $n \geq 1$, the sum of the first $n$ even integers is $n^2 + n$

That is, $2*1 + 2*2 + \ldots + 2*n = n^2 + n$

Now consider the sum of the first $n+1$ even integers. This sum can be written as

\[ 2*1 + 2*2 + \ldots + 2*n + 2*(n+1) \]

\[ = n^2 + n + 2*(n+1) \quad \text{by the inductive hypothesis} \]

\[ = n^2 + 2*n + 1 + n + 1 \quad \text{rearranging} \]

\[ = (n+1)^2 + n + 1 \]

Thus the equation holds for $n+1$, and thus it holds for all $n \geq 1$
Marking Question 2:

Proof by induction involves a base case, an assumption that the statement is true for some \( n \), and a subsequent proof that that statement is true for \( n+1 \).

If a student demonstrates an understanding of the structure of inductive proof, they should get at least 6/10 (3 marks for remembering the base case, 3 marks for remembering the inductive part).

The closer they get to a complete proof, the closer to 10 their mark should be.

If their proof is correct but they forget to give a base case, they should get 7/10.

There are other equally valid sequences of steps to get from the assumption about \( n \) to the conclusion about \( n+1 \). For example, a student might start with \((n+1)^2 + (n+1)\), expand that out, reorganize the terms, then apply the inductive assumption and arrive at the desired summation.
Question 3 (10 marks)

Let $A$ be a finite set of integers, with $|A| = n$. Consider the following relation based on $A$.

$$R = \{ (a, b) \mid a \in A, b \in A, \text{ and } a \neq b \}$$

What is $|R|$? Explain your answer.

Solution:

$$|R| = n*(n-1)$$

Each of the $n$ integers in $A$ can occur in the first position in an ordered pair, and it can be paired with all the other integers in $A$ (of which there are $n-1$). Thus the total number of ordered pairs in $R$ is $n*(n-1)$

Marking Question 3:

If a student demonstrates that they understand that a relation is a set of ordered pairs and that $|R|$ refers to the cardinality of $R$, they should get at least 5/10.

If a student gives a correct example (eg works it out for $A = \{1,2,3\}$) but does not give a general solution, they should get 5/10

If they give the correct answer but don’t explain it, give 7/10

If they give a wrong answer with no explanation, give 4/10

If they give a wrong answer with some explanation, give between 5/10 and 7/10
**Question 4 (10 marks)**

Let $a$ and $b$ be **positive integers** in base-2 notation. Assume that both are represented by exactly $k$ bits (for example, if $a = 13$, $b = 25$, and $k = 5$, the representation of $a$ and $b$ would be $a = 01101$, $b = 11001$)

Explain clearly how we can use the representations of $a$ and $b$ to determine whether $a < b$, $a = b$, or $a > b$

You may wish to use a notation such as “$a[i]$” to refer to the $i^{\text{th}}$ bit of $a$.

**Solution:**

We can compare the bits from left to right. As soon as we find an $i$ where $a[i]$ and $b[i]$ are different, we can decide which of $a$ and $b$ is greater (it is the one with the “1” in the $i^{\text{th}}$ position). If all the bits are the same, then $a$ and $b$ are equal.

More formally, assume that $a[k-1]$ is the leftmost bit of $a$, and $b[k-1]$ is the leftmost bit of $b$.

$i = k-1$
while $i >= 0$:
  if $a[i] == b[i]$:
    $i = i-1$
  else if $a[i] == 1$:
    print “$a > b$”  # or return “$a > b$”, or some similar action
    exit
  else:
    print “$b > a$”  #or return “$a < b$”, or something similar
    exit
print “$a == b$”  #or return “$a==b$”, or something similar
Marking Question 4:

Students are NOT required to write an algorithm as shown above. A clear verbal description of the process, as given in the first paragraph of my solution, is sufficient.

In my solution I indexed the bits so that $a[k-1]$ is the leftmost bit. It is equally valid to index the bits so that $a[0]$ is the leftmost bit, or to use index values from 1 to $k$ rather than 0 to $k-1$.

The key idea to look for is checking the bits from left to right, and finding the first bit where $a$ and $b$ are different. If the student gets that idea correct, they should get at least 7/10.
Question 5 (10 marks)

Let \( x \) and \( y \) be elements of \( \mathbb{Z}_b \), where \( b \) is an integer \( \geq 2 \). When we add \( x \) and \( y \) in base-\( b \) arithmetic, there may or may not be a “carry”. Explain why, if there is a carry, it cannot be greater than 1.

Solution:

\[
0 \leq x < b \text{ and } 0 \leq y < b \quad (\text{since they are both elements of } \mathbb{Z}_b)
\]

Thus \( 0 \leq x+y < 2b \)

Since \( x+y < 2b \), its representation in base \( b \) must contain at most 2 digits (since anything with 3 digits in base-\( b \) is \( \geq b^2 \), which is \( \geq 2b \))

Now if \( b = 2 \), the only possible non-zero carry is 1, since that is the only non-zero value in base-2 notation.

If \( b \geq 3 \), suppose the carry into the left column is \( \geq 2 \). But that would represent a value of at least \( 2b \), and we know \( x+y < 2b \)

Thus the carry, if there is one, cannot be greater than 1
Marking Question 5:

The key point here is that since both $x$ and $y$ are $< b$, their sum must be $< 2\times b$, so if the sum is a 2-digit number, the carry cannot be $> 1$.

If a student understands what the question is asking (ie understands $\mathbb{Z}_b$ and base-b) they should get at least 5/10

If they also see that the sum must be $< 2\times b$, they should get at least 7/10

If they give a good explanation of why this means the carry digit is either 0 or 1, they should get between 7/10 and 10/10
This page is for rough work or overflow.