This is a closed book test. You may not refer to any resources.

This is a 50 minute test.

Please write your answers in ink. Pencil answers will be marked, but will not be reconsidered after the test papers have been returned.

The test will be marked out of 50.

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Question 1 (10 marks)

a) Consider a connected graph \( G \) with no cut edges. Show that \( G \) has no vertices with degree one.

<Definition: a cut edge is one that, if removed, increases the number of components of the graph.>

Solution: Let \( G \) be a connected graph with no cut edges, and suppose \( G \) has a vertex \( x \) with degree one. This means \( x \) has exactly one neighbour \( y \), and \((x,y)\) is an edge of \( G \). In \( G - e \), \( x \) has degree zero, so it is no longer connected to \( y \). Therefore \( e \) is a cut edge of \( G \). CONTRADICTION.

Therefore \( G \) has no cut edges.

b) Show an example of a connected graph that has at least one cut edge, but has no vertices of degree one.

Solution: (one of infinitely many possibilities). In this graph, edge \( e \) is a cut edge.
Question 2 (10 marks)

Consider the following graph.

a) Find a clique of largest possible size for this graph.

Solution: \{b, c, e, f\} is a clique of size 4. There is no clique of size 5 because that would require five vertices each having degree \( \geq 4 \) and there are only three such vertices.

b) Find an independent set of largest possible size for this graph.

Solution: \{a, c, d\} is an independent set of size 3. There is no independent set of size 4 because at most one of \{b, c, e, f\} can be in any independent set, and that only leaves a and d as other usable vertices.

c) Find a spanning subtree of this graph. You may wish to draw your answer.

Solution: one of many possible choices:
Question 3 (10 marks)

Consider the graph $H_n$ whose vertices are elements of the set $\{1,2,...,n\}$. In $H_n$, there is an edge between two vertices $u,v$ if and only if $u+v$ is even.

For example, here is $H_4$:

![Graph](image)

a) How many edges are there in $H_n$?

Solution: $u+v$ is even iff $u$ and $v$ are both even, or $u$ and $v$ are both odd. Thus all odd vertices are neighbours and all even vertices are neighbours. In other words, $H$ consists of two disjoint complete subgraphs, one containing all the even vertices and the other containing all the odd vertices.

If $n$ is even, each of these has $\frac{n}{2}$ vertices, and since they are complete each has $\binom{n/2}{2}$ edges, so the total number of edges is $2 \times \frac{n}{2} \times \left(\frac{n}{2} - 1\right)$ (which can be simplified or left like this).

If $n$ is odd, $G$ has $\frac{n+1}{2}$ odd vertices and $\frac{n-1}{2}$ even vertices. The two complete subgraphs have $\frac{n+1}{2} \times \left(\frac{n+1}{2} - 1\right)$ and $\frac{n-1}{2} \times \left(\frac{n-1}{2} - 1\right)$ edges respectively, so the number of edges in $H_n$ is the sum of these.
An alternative solution is to give a recurrence relation: Let $E_n$ be the number of edges of $H_n$.

\[ E_1 = 0 \]
\[ E_2 = 0 \]

\[ E_n = \begin{cases} 
  E_{n-1} + \frac{n-1}{2} & \text{if } n \text{ is odd} \\
  E_{n-1} + \frac{n-2}{2} & \text{if } n \text{ is even}
\end{cases} \]

This solution is much more concise but much harder to find!

b) For what values of $n$ is the graph $H_n$ connected?

Solution: $H_n$ is only connected for $n = 1$. For all larger values of $n$ there are even vertices and odd vertices and none of the even vertices are connected to any of the odd vertices.
Question 4 (10 marks)

Consider a graph $G$ consisting of two connected components, each of which has no cycles. Consider adding to $G$ one edge whose two vertices are from different components. Show that the resulting graph is a tree.

Solution: Let $G'$ be the graph that results from adding edge $(x,y)$ where $x$ is in one component of $G$ and $y$ is in the other.

Claim: $G'$ is connected: Let $w$ and $z$ be any vertices of $G'$. If they are both in the same connected component of $G$, then they are connected by definition. If they are in different components of $G$, assume WLOG $w$ is in the same component as $x$ and $z$ is in the same component as $y$. $G$ contains a path from $w$ to $x$ and a path from $y$ to $z$. Combining these with the new edge $(x,y)$ gives a path from $w$ to $z$ in $G'$, so they are connected in $G'$.

Claim: $G'$ has no cycles: Suppose $G'$ has a cycle. This cycle must contain the edge $(x,y)$ because $G$ has no cycles. This implies that the rest of the cycle consists of a path joining $x$ and $y$ in $G$. But $x$ and $y$ are not connected in $G$. CONTRADICTION.

Therefore $G'$ has no cycles.

Thus $G'$ is connected and has no cycles. $G'$ is a tree.
Question 5 (10 marks)

What is the chromatic number of the graph below? Justify your answer.

<Definition: the chromatic number of a graph is the smallest number of colours needed for a legal colouring of the vertices (ie a colouring in which no adjacent vertices have the same colour.> 

Solution: The chromatic number is 3. Here is a legal 3-colouring:

The graph cannot be 2-coloured because it contains $K_3$ as a subgraph