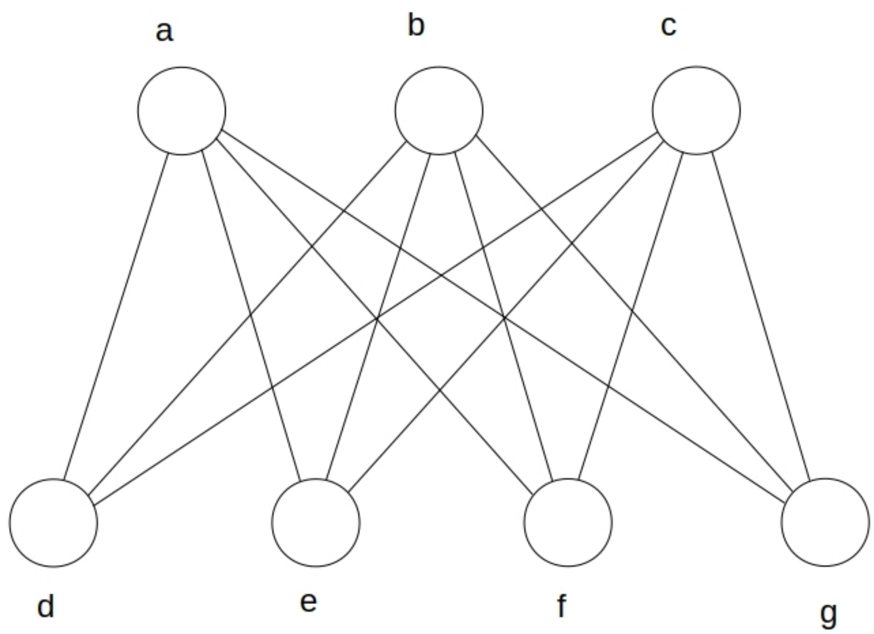
Question 1: (10 Marks)



Does this graph contain a cycle that includes all 7 vertices? If so, list the vertices in the order they appear in the cycle. If not, explain why there is no such cycle.

Solution:

It does not.

Suppose there were such a cycle. Due to the structure of the graph, the vertices that follow the vertices in {d,e,f,g} in the cycle must all be in the set {a,b,c} and they must all be different. The pigeon-hole principle shows that this is not possible.

Therefore there is no such cycle.

Marking: Students do not have to refer to the pigeon-hole principle by name, but their argument should express the idea that any Hamilton Cycle (which they also don't have to name) would have to alternate back and forth between the two sets {a,b,c} and {d,e,f,g}, which is not possible.

There are other ways to prove the result. For example, a student could point out that this is a bipartite graph, and bipartite graphs are all 2-colourable, and 2-colourable graphs cannot contain odd-length cycles ... this is a correct argument although it uses a lot of information that we didn't cover.

For a sound argument that shows there is no cycle that includes all vertices:	**10/10**
For a poorly-expressed argument that seems to be on the right track (such as "there are too many vertices in the bottom row" or "you run out of vertices")	**7/10**
For a solution that contains no proof but does show understanding of what a cycle is	**5/10**
For an argument that doesn't show understanding of what a cycle is	**3/10**
For trying	**1/10**

Question 2 : (15 marks)

Let $S = \{1, 2, 3\}$ and let F be the set of all functions from S to S.

Note that F contains functions which are bijections as well as functions that are not.

We can create a graph in which *each vertex represents a function in F*. The edges of the graph are determined as follows:

The vertex for function f is adjacent to the vertex for function g if and only if there is exactly one difference between the functions.

More formally, the vertices representing functions f and g are adjacent in the graph if and only if there is exactly one $i \in S$ for which $f(i) \neq g(i)$

For example, consider the functions f, g and h shown in this table

i	$f(i)$	$g(i)$	$h(i)$
1	2	1	2
2	3	3	1
3	2	2	1

The vertices representing f and g are adjacent because $f(1) \neq g(1)$ and there is no other i such that $f(i) \neq g(i)$. The vertices representing f and h are not adjacent because there are two i such that $f(i) \neq h(i)$. The vertices representing g and h are not adjacent because $g(i) \neq h(i) \quad \forall\ i$

Question continues on the next page

(a) [5 marks] Let f be the function from the example above. List ALL the functions whose vertices are adjacent to the vertex for function f

Solution: I will use [a b c] to represent the function j in which j(1) = a, j(2) = b, j(3) = c. Using this notation, f = [2 3 2]

The functions whose vertices are adjacent to the vertex for f are ...

[1 3 2]   [3 3 2]   [2 1 2]   [2 2 2]   [2 3 1]   [2 3 3]

Marking: deduct 1 mark for each function missed (but don't go below 0)
deduct 1 mark for each incorrect function included (don't go below 0)

Students are not required to use this notation

(b) [5 marks] Show that this graph is connected. You are not required to draw the graph (but you may if you want to). Hint: show there is a path from every function to the function p, **where** $p(1) = p(2) = p(3) = 1$

Solution: Using the same notation, let [a b c] be any function other than [1 1 1]

One step at a time, we can eliminate the differences between [a b c] and [1 1 1], and each step corresponds to an edge in the graph.

If $a \neq 1$
    ([a b c], [1 b c]) is an edge.
Else
    we know [a b c] = [1 b c]
If $b \neq 1$
    ([1 b c], [1 1 c]) is an edge.
Else
    we know [1 b c] = [1 1 c]
If $c \neq 1$
    ([1 1 c], [1 1 1]) is an edge.
Else
    we know [1 1 c] = [1 1 1]

Thus there is a path from [a b c] to [1 1 1]

Thus for any two functions [a b c] and [d e f], we can follow a path from [a b c] to [1 1 1] and then follow a path from [1 1 1] to [d e f]. Thus there is a walk/path from [a b c] to [d e f]

Thus the graph is connected.

Marking:

For any sound demonstration that there is a path between any two vertices	**5/5**
For a weak or partial explanation that shows some understanding of the meaning of connectivity	**3/5**
For trying	**1/5**

(c) [5 marks] Is this graph Eulerian? Explain your answer. You may use the fact that the graph is connected, even if you did not answer part (b)

Solution:

Part (a) generalizes to show that every vertices has 6 neighbours, so all degrees are even

([a b c] has 2 neighbours that change a, 2 neighbours that change b and 2 neighbours that change c)

Part (b) showed that the graph is connected.

The graph is connected and all degrees are even. Therefore it is Eulerian.

Marking:

For a sound argument that shows the graph is Eulerian	**5/5**
For a poor argument that shows understanding of the term Eulerian	**3/5**
For trying	**1/5**

Question 3: (10 Marks)

Let G be a connected graph on ≥ 3 vertices. Let P_1 and P_2 be paths in G. We say P_1 is *contained in* P_2 if all edges in P_1 are also in P_2

Show that "*contained in*" *is* a partial ordering of the set of all paths in G.

Solution:

Reflexive: ***Let P be a path. P contains all of its own edges, so P is contained in P***

Thus "contained in" is reflexive

Transitive: ***Suppose P_1 is contained in P_2, and P_2 is contained in P_3***

Let e be an edge in P_1. We know all edges in P_1 are in P_2, so e is in P_2. We know all edges in P_2 are in P_3, so e is in P_3. Therefore P_1 is contained in P_3

Thus "contained in" is transitive

Anti-Symmetry: ***Suppose P_1 is contained in P_2 , and P_2 is contained in P_1. The the set of edges of P_1 is a subset of the set of edges of P_2, and the set of edges of P_2 is a subset of the set of edges of P_1.***

We know two sets are subsets of each other iff they are equal.

$\Rightarrow$ ***the set of edges of P_1 = the set of edges of P_2***

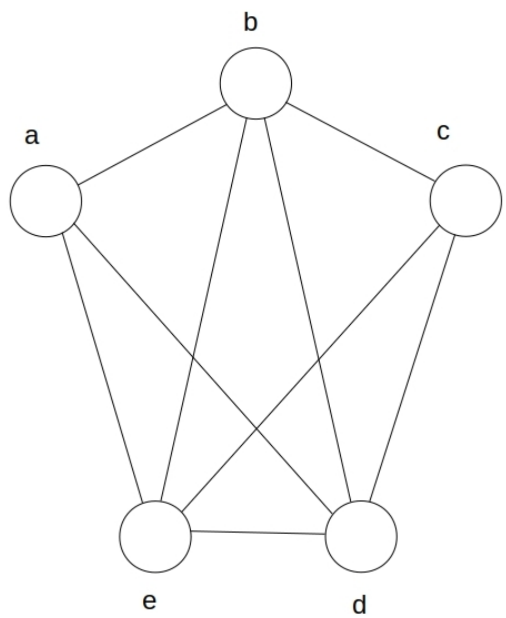
$\Rightarrow P_1 = P_2$

Thus "contained in" is anti-symmetric

Marking : 3 marks for reflexivity and transitivity, 4 marks for anti-symmetry with part marks for incomplete answers.

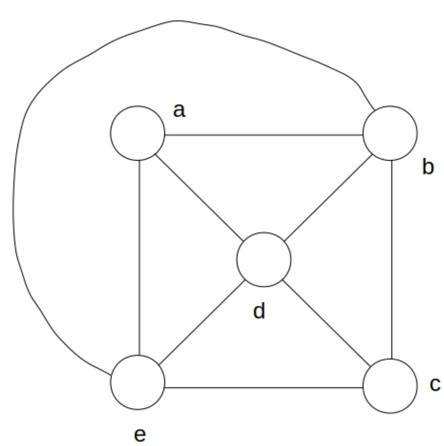
Question 4 : (10 Marks)

Consider the graph that results from deleting one edge from K_5 :



(a) [7 marks]If this graph is planar, show a planar drawing of it. If it is not planar, explain why not.

Solution:



(b) [3 marks] What is the colouring number of this graph? Explain.

Solution:

Vertices {a,b,d,e} form a K_4 *({b,c,d,e} also form a* K_4*) so it needs at least 4 colours. The graph is planar so it needs at most 4 colours. Therefore the colouring number is 4.*

Marking: (a)

For any correct planar drawing	**7/7**
For a correct planar drawing without vertex labels	**5/7**
For an incorrect planar drawing (such as one with edges missing)	**3/7**
For stating "non-planar" and attempting a proof	**3/7**
For trying	**1/7**

(b)

For giving the correct answer and some justification	**3/3**
For giving the correct answer with no justification	**2/3**
For giving the wrong answer, but showing understanding of colouring	**2/3**
For trying	**1/3**

Question 5 : (5 Marks)

Let G, H and J be graphs with $G \cong H$ and $H \cong J$.

(Recall that $\cong$ means "is isomorphic to")

Either prove that $G \cong J$ or give a counterexample.

Solution:

Let f_1 be an isomorphism from G to H, and let f_2 be an isomorphism from H to J

Thus for $x, y \in G, (x, y) \in E_G$ iff $(f_1(x), f_1(y)) \in E_H$
and $(f_1(x), f_1(y)) \in E_H$ iff $(f_2(f_1(x)), f_2(f_1(y))) \in E_J$

Therefore $(x, y) \in E_G$ iff $(f_2(f_1(x)), f_2(f_1(y))) \in E_J$

Therefore $f_2 \circ f_1$ is an adjacency-preserving bijection between G and J

Therefore $G \cong J$

Marking:

For a sound argument showing that an isomorphism exists between G and J	**5/5**
For an incomplete or incorrect answer that shows understanding of the word "isomorphism"	**3/5**
For trying	**1/5**