CISC-203*
Test #2
October 19, 2017

Student Number (Required) ______________________

Name (Optional) ________________________________

This is a closed book test. You may not refer to any resources.

This is a 50 minute test.

Please write your answers in ink. Pencil answers will be marked, but will not be reconsidered after the test papers have been returned.

The test will be marked out of 50.

| Question 1 | /10 |
| Question 2 | /10 |
| Question 3 | /10 |
| Question 4 | /10 |
| Question 5 | /10 |
| **TOTAL**  | **/50** |

“True education flowers at the point when delight falls in love with responsibility.“

Happy Birthday to Philip Pullman
Question 1 (10 marks)

Let S be the set of all permutations of \{1, 2, \ldots, n\}.

Let \( \pi \) be a permutation randomly chosen from the set S, where the probability of each permutation being chosen is \( P(\pi) = \frac{1}{n!} \ \forall \pi \in S \) (in other words, each permutation is equally likely to be chosen.)

(a) (3 marks) What is the probability that \( \pi(1) = 1 \)?

Solution: When \( \pi(1) = 1 \) there are \((n-1)!\) ways to arrange the other elements, so there are \((n-1)!\) permutations in which \( \pi(1) = 1 \). Therefore the probability of \( \pi(1) = 1 \) is
\[
\frac{(n - 1)!}{n!} = \frac{1}{n}
\]

Marking: 3/3 for getting \( \frac{1}{n} \)

2/3 for having the right idea but getting the answer wrong

1/3 for trying but not really knowing what to do

0/3 for not trying or having no idea what was expected
(b) (3 marks) What is the probability that $\pi(1) = 1$ and $\pi(2) = 2$?

Solution: When $\pi(1) = 1$ and $\pi(2) = 2$, there are $(n-2)!$ ways to arrange the other elements, so there are $(n-2)!$ permutations in which $\pi(1) = 1$ and $\pi(2) = 2$. Therefore the probability of $\pi(1) = 1$ and $\pi(2) = 2$ is given by

$$P(\pi(1) = 1 \text{ and } \pi(2) = 2) = \frac{(n-2)!}{n!} = \frac{1}{(n-1) \times n}$$

Marking: same as for (a) ... but if they made the same error in both parts (for example, forgetting that the number of permutations on $n$ elements is $n!$) they should not be penalized twice for the same error.
(c) (4 marks) What is the probability that $\pi(2) = 2$, given that $\pi(1) = 1$?

Solution:

$$P(\pi(2) = 2 | \pi(1) = 1) = \frac{P(\pi(1) = 1 \text{ and } \pi(2) = 2)}{P(\pi(1) = 1)}$$

$$= \frac{\frac{1}{(n-1) \times n}}{\frac{1}{n}}$$

$$= \frac{n}{(n - 1) \times n}$$

$$= \frac{1}{n - 1}$$

Marking: 4/4 for computing the conditional probability correctly, based on their answers to (a) and (b)

(note – it is acceptable to give an argument such as “if $\pi(1) = 1$, then there are $n-1$ places 2 could go ... so the probability that $\pi(2) = 2$ must be $\frac{1}{n - 1}$

3/4 for showing they understand conditional probability but not being able to apply it

2/4 for knowing there is such a thing as conditional probability but not knowing what it is

1/4 for trying

0/4 for not trying
Question 2 (10 marks)

Let $S = \{1,2,3,4\}$ and let $P(s) = \frac{1}{4}$ $\forall s \in S$

Let $A = \{1,2\}$, $B = \{2,3\}$, and $C = \{3,4\}$ be events on the sample space $(S,P)$

(a) (3 marks) Are $A$ and $B$ independent? Explain your answer

Solution: $A$ and $B$ are independent iff $P(A \cap B) = P(A) \times P(B)$

$A \cap B = 2$ so $P(A \cap B) = P(2) = \frac{1}{4}$

$P(A) = P(B) = \frac{2}{4} = \frac{1}{2}$ so $P(A) \times P(B) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

Thus $A$ and $B$ are independent

Marking 3/3 for getting it right (and showing how they got there) 2/3 for showing they knew what to do, but they got it wrong 1/3 for stating “independent” but not explaining 0/3 for not trying

(b) (3 marks) Are $B$ and $C$ independent? Explain your answer

Solution: basically the same as for (a)

Marking same as for (a), but please do not penalize twice for the same error
(c) (4 marks) Are A and C independent? Explain your answer

Solution: Basically the same as for (a) and (b) except that $A \cap C = \emptyset$ so $P(A \cap C) = 0$

Since $P(A) \times P(C) = \frac{1}{4}$ we conclude that A and C are not independent

Marking  
4/4 for getting it right and explaining why  
3/4 for showing they knew what to do, but they got it wrong  
2/4 for stating “not independent” but not explaining  
1/4 for trying  
0/4 for not trying
Question 3 (10 marks)

Let $S$ = the set of outcomes of tossing 3 fair coins

Define random variables $X$, $Y$ and $Z$ on $(S,P)$ as

$X(s) = 10$ if the number of Heads showing is $\leq 2$
$X(s) = 20$ if the number of Heads showing $= 3$

$Y(s) = 1$ if the number of Heads showing is odd
$Y(s) = 2$ if the number of Heads showing is even

$Z(s) = X(s) + Y(s)$
(a) (3 marks) What is $E(X)$ (i.e. the expected value of $X$)? Show how you found the answer.

Solution: There are 8 possible outcomes. Since the coins are fair, each outcome has probability $= \frac{1}{8}$

<table>
<thead>
<tr>
<th>$s$</th>
<th>$X(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>HHH</td>
<td>20</td>
</tr>
<tr>
<td>HHT</td>
<td>10</td>
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<td>10</td>
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<td>TTT</td>
<td>10</td>
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$E(X) = \sum_{a \in V} a \cdot P(X = a)$ or $= \sum_{s \in S} X(s) \cdot P(s)$ (equivalent definitions)

So $E(X) = 20 \cdot \frac{1}{8} + 10 \cdot \frac{7}{8} = \frac{90}{8}$

Marking: 3/3 for getting it right and showing their work. They don’t need to state the definition of $E(X)$
2/3 for knowing what to do but getting the answer wrong
1/3 for getting the right answer but not showing how they got it
1/3 for trying
0/3 for not trying

(Question 3 continues on the next page)
(b) (3 marks) What is E(Y) ? Show how you found the answer.

<table>
<thead>
<tr>
<th>S</th>
<th>Y(s)</th>
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<tbody>
<tr>
<td>HHH</td>
<td>1</td>
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<td>HHT</td>
<td>2</td>
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<tr>
<td>TTT</td>
<td>2</td>
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</tbody>
</table>

Using the same method as in (a), we get $E(Y) = 1 \cdot \frac{4}{8} + 2 \cdot \frac{4}{8} = \frac{12}{8} = \frac{3}{2}$

Marking: same as for (a) but please don’t penalize twice for the same mistake
(c) (2 marks) What is $E(Z)$? Show how you found the answer.

**Solution:** $E(Z) = E(X) + E(Y)$ so $E(Z) = \frac{90}{8} + \frac{12}{8} = \frac{102}{8}$

**Marking:**
- 2/2 for getting it right and showing their work
- 1/2 for getting it right and not showing their work
- 1/2 for knowing what to do but getting it wrong
- 0.5/2 for trying
- 0/2 for not trying
(d) (2 marks) Are X and Y independent random variables? Explain your answer.

Solution: X and Y are independent if

\[ \forall a, b \quad P(X = a \text{ and } Y = b) = P(X = a) \times P(Y = b) \]

Let \( a = 20 \) and \( b = 2 \)

There is no outcome \( s \) in \( S \) that gives \( X(s) = 20 \) and \( Y(s) = 2 \), so

\[ P(X = 20 \text{ and } Y = 2) = 0 \]

But \( P(X = 20) = \frac{1}{8} \) and \( P(Y = 2) = 1/2 \) so

\[ P(X = 20) \times P(Y = 2) = \frac{1}{16} \neq 0 \]

Thus \( P(X = a \text{ and } Y = b) = P(X = a) \times P(Y = b) \) does not hold for \( a = 20 \), \( b = 2 \)

Therefore X and Y are not independent

Marking: 

2/2 for “not independent” with explanation
1/2 for “not independent” without explanation
1/2 for “independent” with (incorrect) explanation
1/2 for showing they know what “independence” means, without being able to apply it
0/2 for not trying
Question 4 : (10 Marks)

Four players play the following game. Four cards are shuffled and dealt face down. One card is black and the other three are red – but of course they all look the same when they are face down. The players take turns to choose and turn over one of the face down cards (Player 1 goes first, Player 2 goes second, etc.) Once a card is turned face up, it stays face up. Whoever turns over the black card wins the game.

If Player 1 wins, she wins $1
If Player 2 wins, she wins $2
If Player 3 wins, she wins $3
If Player 4 wins, she wins $4

If you were playing this game, which player would you choose to be? Explain your reasoning carefully.
Solution:

The probability of each player winning is $\frac{1}{4}$. We can work this out using conditional probability or we can simply argue that no matter which cards the players pick, $\frac{1}{4}$ of all the card arrangements will place the black card in the spot Player 1 picks, $\frac{1}{4}$ will place the black card in the spot Player 2 picks, etc.

Since each player has the same probability of winning, I would choose to be Player 4 – the payoff is better than for any of the other players.

Marking: 10/10 for a reasonable argument that all players have the same probability of winning, and concluding that it is best to be Player 4
7/10 for giving a faulty explanation for choosing Player 4
6/10 for giving a faulty explanation for choosing a different player
5/10 for choosing to be Player 4, without explaining
3/10 for choosing to be a different player, without explaining
1/10 for trying
0/10 for not trying
Question 5: (10 Marks)

(a) (5 marks) Suppose we are doing modular arithmetic on \( \mathbb{Z}_7 \)

If \( 4 \otimes 3 = 2 \otimes x \), what is \( x \)? Explain your reasoning.

Solution: In \( \mathbb{Z}_7 \), \( 4 \otimes 3 = (4 \times 3) \% 7 = 12 \% 7 = 5 \)

So we need to find \( x \) such that \( 2 \otimes x = 5 \) in \( \mathbb{Z}_7 \)

A bit of thought or experimentation reveals that \( x = 6 \)

Marking: 5/5 for getting it right and showing their work
3/5 for understanding \( \otimes \) but getting the answer wrong
2/5 for getting 6 but not explaining
1/5 for trying
0/5 for not trying
(b) (5 marks) Suppose we are doing modular arithmetic on $\mathbb{Z}_n$, where $n \geq 2$

Prove that for every $a \in \mathbb{Z}_n$, there exists $b \in \mathbb{Z}_n$ such that $a \oplus b = 1$

Solution:

If $a = 0$, then $b = 1$
If $a = 1$, then $b = 0$
If $a > 1$, let $b = n + 1 - a$. Observe that $a < n$, so $1 \leq b \leq n - 1$,
so $b \in \mathbb{Z}_n$

$a \oplus b = (a + b) \mod n = (a + n + 1 - a) \mod n = (n + 1) \mod n = 1$

Marking: 5/5 for a complete answer
3/5 for an answer that does not deal correctly with $a=0$ and $a=1$
2/5 for an incorrect answer that shows understanding of $\oplus$
1/5 for trying
0/5 for not trying